Non-relativistic limit of the MIT bag model

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The MIT bag model

This model was introduced by Chodos, Jaffe, Johnson, Thorn, and Weisskopf, physicists from the MIT, in order to understand the confinement of the quarks/anti-quarks inside the hadrons.

a. Quarks/anti-quarks are elementary particles.
b. Hadrons are particles composed by quarks and anti-quarks.
c. Note that no isolated quark has been observed yet.
d. They only consider bosonic hadrons, i.e. pairs quark/anti-quark.
e. They assume that the pair is perfectly confined in $\Omega \subset \mathbb{R}^3$.
f. The quarks are relativistic particles of spin $\frac{1}{2}$.

The region of space Ω where the quarks live is called the bag. It is assumed to be bounded and smooth.

The Dirac operator

We consider the differential operator of order 1, acting on $L^2(\Omega, \mathbb{C})^4$, defined by:

$$H = -i\alpha \cdot \nabla + m\beta\,.$$

m is the mass of the quark or of the anti-quark.
From a mathematical point of view, m can be positive or negative.

The Dirac operator

$$H = \alpha \cdot D + m\beta\,, \qquad D = -i\nabla\,.$$

We have $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. The α_k and β are 4×4 Hermitian and unitary matrices.

$$\beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \ \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \text{ for } k = 1, 2, 3\,.$$

The Pauli matrices σ_1, σ_2 and σ_3 are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The symbol $\alpha \cdot X$ denotes $\sum_{j=1}^{3} \alpha_j X_j$ for any $X = (X_1, X_2, X_3)$.

The boundary condition

Now, we must translate the perfect confinement condition mathematically. It is a boundary condition. On $\partial\Omega$, we impose that the wavefunctions satisfy

$$\mathcal{B}\psi := -i\beta(\alpha \cdot \mathbf{n})\psi = \psi \, ,$$

where $\mathbf{n}$ is the outward pointing normal to the boundary.
This condition is chosen to have no normal quantum current. We impose that $\psi_{|\partial\Omega}$ is an eigenvector of $\mathcal{B}$. Note that $\mathcal{B}$ is Hermitian and that $\mathcal{B}^2 = 1_4$.

If one wants to consider the inward pointing normal situation, we can just change the boundary condition into $\mathcal{B}\psi = -\psi$ which also implies a perfect confinement.

Definition of the MIT bag operator

Definition

The MIT bag Dirac operator $(H_m^\Omega, \mathcal{D}(H_m^\Omega))$ is defined on the domain

$$\mathrm{Dom}(H_m^\Omega) = \{\psi \in H^1(\Omega, \mathbb{C})^4 \ : \ \mathcal{B}\psi = \psi \text{ on } \Gamma\}\,,$$

by $H_m^\Omega \psi = H\psi$ for all $\psi \in \mathrm{Dom}(H_m^\Omega)$.

Note that the trace is well-defined by a classical trace theorem.

Chirality matrix and negative mass

We introduce the chirality matrix

$$\gamma_5 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} .$$

We notice that

$$\gamma_5 \left(\alpha \cdot D - m\beta \right) \gamma_5 = \alpha \cdot D + m\beta \,, \qquad \gamma_5 \mathcal{B} \gamma_5 = -\mathcal{B} \,.$$

Thus, $\alpha \cdot D - m\beta$ with boundary condition $\mathcal{B}\psi = \psi$ is unitarily equivalent to $\alpha \cdot D + m\beta$ with boundary condition $\mathcal{B}\psi = -\psi$.
In other words, if we allow the mass to be negative, the model also describes the case of the boundary condition $\mathcal{B}\psi = -\psi$.
The spectral behavior of the MIT bag model strongly depends on the sign of m (or equivalently: the orientation of the normal, the sign of the boundary condition).

Properties of the operator related to self-adjointness

Theorem

i. $(H, \mathrm{Dom}(H))$ *is a self-adjoint operator with compact resolvent.*

ii. *We denote by* $(\mu_n(m))_{n\geq 1} \subset \mathbb{R}_+^*$ *the eigenvalues of* $|H|$. *The spectrum of* H *is symmetric with respect to* 0 *(with multiplicity) and*

$$\mathrm{sp}(H) = \{\pm\mu_n(m),\ n \geq 1\}\,.$$

iii. *Each eigenvalue* $\mu_n(m)$ *has pair multiplicity.*

iv. *For each* $\psi \in \mathrm{Dom}(H)$, *we have*

$$\|H\psi\|^2_{L^2(\Omega)} = \|\alpha\cdot\nabla\psi\|^2_{L^2(\Omega)} + m\|\psi\|^2_{L^2(\partial\Omega)} + m^2\|\psi\|^2_{L^2(\Omega)}\,,$$

$$\|\alpha\cdot\nabla\psi\|^2_{L^2(\Omega)} = \|\nabla\psi\|^2_{L^2(\Omega)} + \frac{1}{2}\int_{\partial\Omega}\kappa|\psi|^2 ds\,.$$

Let us discuss the proof of the formula for the square of H:
$\forall\psi \in \mathrm{Dom}(H)$

$$\|H\psi\|^2_{L^2(\Omega)} = m^2\|\psi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega)} + \int_{\partial\Omega}\left(\frac{\kappa}{2} + m\right)|\psi|^2 ds\,.$$

Lemma

For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, *we have*

$$(\alpha\cdot\mathbf{x})(\alpha\cdot\mathbf{y}) = (\mathbf{x}\cdot\mathbf{y})1_4 + i\gamma_5\alpha\cdot(\mathbf{x}\times\mathbf{y})\,,$$
$$\beta(\alpha\cdot\mathbf{x}) = -(\alpha\cdot\mathbf{x})\beta\,,\quad \beta\gamma_5 = -\gamma_5\beta\,,$$
$$\gamma_5(\alpha\cdot\mathbf{x}) = (\alpha\cdot\mathbf{x})\gamma_5\,.$$

Lemma (Mean curvature as commutator)

$$[\alpha\cdot(\mathbf{n}\times D), \mathcal{B}] = -\kappa\gamma_5\mathcal{B}\,.$$

We have

$$\|H\psi\|^2_{L^2(\Omega)} = \langle \alpha\cdot D\psi, \alpha\cdot D\psi\rangle_\Omega + m^2\langle \beta\psi, \beta\psi\rangle_\Omega + 2m\,\mathrm{Re}\langle \beta\psi, \alpha\cdot D\psi\rangle_\Omega\,.$$

By using that α anticommutes with β and an integration by parts

$$2\,\mathrm{Re}\langle \beta\psi, \alpha\cdot D\psi\rangle_\Omega = \langle i\alpha\cdot\mathbf{n}\beta\psi, \psi\rangle_{\partial\Omega} = \langle -i\beta\alpha\cdot\mathbf{n}\psi, \psi\rangle_{\partial\Omega} = \|\psi\|^2_{L^2(\partial\Omega)}\,.$$

Since β is unitary,

$$\|H\psi\|^2_{L^2(\Omega)} = \|\alpha\cdot D\psi\|^2_{L^2(\Omega)} + m^2\|\psi\|^2_{L^2(\Omega)} + m\|\psi\|^2_{L^2(\partial\Omega)}\,.$$

Assume that $\psi \in H^2(\Omega)$. By the Green-Riemann formula

$$\begin{aligned}\langle \alpha\cdot D\psi, \alpha\cdot D\psi\rangle_\Omega &= \langle \psi, (\alpha\cdot D)^2\psi\rangle_\Omega + \langle (-i\alpha\cdot\mathbf{n})\psi, \alpha\cdot D\psi\rangle_{\partial\Omega}\\ &= \langle D\psi, D\psi\rangle_\Omega + i\langle \psi, \left((\alpha\cdot\mathbf{n})(\alpha\cdot D) - (\mathbf{n}\cdot D)\right)\psi\rangle_{\partial\Omega}\end{aligned}$$

($(\alpha\cdot D)^2 = 1_4 D^2$ and by another integration by parts)
$H^2(\Omega)$ dense in $H^1(\Omega)$, thus it holds for any $u \in \mathrm{Dom}(H)$.

Relation with shell interactions

In [1] we prove that $H + V_{es}$ generates confinement w.r.t. Γ for $\lambda_e^2 - \lambda_s^2 = -4$, where

$$V_{es}\psi = \frac{1}{2}(\lambda_e + \lambda_s\beta)(\psi_+ + \psi_-)d\Gamma\,,$$

$\lambda_e, \lambda_s \in \mathbb{R}$, $\psi_\pm$ are the non-tangential boundary values of ψ on Γ and $d\Gamma$ is the surface measure on Γ.
We know that

$$\ker(H + V_{es} - \mu) \neq 0 \Longleftrightarrow \ker(\lambda_s\beta - \lambda_e + 4C_{\sigma,\mu}) \neq 0\,. \qquad (1)$$

The r.h.s. of (1) is equivalent to the existence of a solution $\psi \in H^1(\Omega, \mathbb{C}^4)$ of the problem $(H - \mu)\psi = 0$ in Ω and $\psi = \frac{i}{2}(\lambda_e - \lambda_s\beta)(\alpha \cdot \mathbf{n})\psi$ on Γ.
When $\lambda_e = 0$ and $\lambda_s = 2$ we recover the MIT bag model.

[1] A., Mas, Vega. Shell interactions for Dirac operators: on the point spectrum and the confinement. *SIAM J. Math. Anal.*, 2015.

Non-relativistic limit: positive mass

From the expression for H^2, when $m \to +\infty$, the operator $H^2 - m^2$ tends, in some sense, towards the Dirichlet Laplacian on Ω. So, it is a non-relativistic limit since it relates the MIT bag model (relativistic particles in a box) to the model for non-relativistic particles in a box.

Theorem

Let $-\Delta^{\mathrm{Dir}}$ be the Laplacian with domain $H^2(\Omega,\mathbb{C}) \cap H^1_0(\Omega,\mathbb{C})$, and let $(\mu_n^{\mathrm{Dir}})_{n\geq 1}$ be the non-decreasing sequence of its eigenvalues. For all $n \geq 1$, we have

$$\mu_n(m) - \left(m + \frac{1}{2m}\mu_n^{\mathrm{Dir}}\right) \underset{m\to+\infty}{=} o\left(\frac{1}{m}\right).$$

Idea of the proof

We work with the operator H^2 appearing previously and determine the asymptotic expansions of its lowest eigenvalues.
For $m > 0$ and $\psi \in \mathrm{D} = \{\psi \in H^1(\Omega, \mathbb{C})^4,\ \psi \in \ker(\mathcal{B} - 1_4) \text{ on } \Gamma\}$, we let

$$Q_m(\psi) = \|\nabla\psi\|^2 + \int_\Gamma \left(m + \frac{\kappa}{2}\right) |\psi|^2 \Gamma\,.$$

For $\psi \in H_0^1(\Omega, \mathbb{C})^4,\ Q_\infty(\psi) = \|\nabla\psi\|^2\,.$

$(\lambda_j(Q_m))_{j\geq 1}$ and $(\lambda_j(Q_\infty))_{j\geq 1} \equiv$ the ordered sequence of eigenvalues related to the operators associated with Q_m and Q_∞.

Proposition

For all $j \geq 1$, *we have* $\lim_{m\to+\infty} \lambda_j(Q_m) = \lambda_j(Q_\infty)\,.$

It is actually possible to describe the next term in the expansion of the first positive eigenvalue.

Theorem

Let $u_1 \in H_0^1(\Omega, \mathbb{C})$ be a L^2-normalized eigenfunction of the Dirichlet Laplacian associated with its lowest eigenvalue μ_1^{Dir}. We have

$$\mu_1(m) - \left(m + \frac{1}{2m}\mu_1^{\mathrm{Dir}} - \frac{1}{2m^2}\int_\Gamma |\partial_{\mathbf{n}} u_1|^2 d\Gamma \right) \underset{m\to+\infty}{=} o\left(\frac{1}{m^2}\right).$$

Remark: This asymptotic expansion of $\mu_1(m)$ coincides with the one of the first eigenvalue of $\sqrt{m^2 - \Delta_{2m}^{\mathrm{Rob}}}$ where $-\Delta_{2m}^{\mathrm{Rob}}$ is the Robin Laplacian of mass $2m$, *i.e.* the operator of $L^2(\Omega, \mathbb{C})$ whose quadratic form is defined for $u \in H^1(\Omega, \mathbb{C})$ by

$$u \longmapsto \int_\Omega |\nabla u|^2 d\mathbf{x} + 2m \int_\Gamma |u|^2 d\Gamma.$$

Non-relativistic limit: negative mass

The boundary is attractive for the eigenfunctions with eigenvalues lying essentially in the Dirac gap $[-|m|, |m|]$ and that their distribution is governed by the operator

$$\mathcal{L}^\Gamma - \frac{\kappa^2}{4} + K\,,$$

where κ and K are the trace and the determinant of the Weingarten map, respectively, and where $\mathcal{L}^\Gamma$ is defined as follows.

Definition

The operator $(\mathcal{L}^\Gamma, \mathrm{Dom}(\mathcal{L}^\Gamma))$ is the self-adjoint operator associated with the quadratic form

$$\mathcal{Q}^\Gamma(\psi) = \int_\Gamma |\nabla_s \psi|^2 d\Gamma\,, \quad \forall \psi \in H^1(\Gamma, \mathbb{C})^4 \cap \ker(\mathcal{B} - 1_4)\,.$$

Theorem

Let $\varepsilon_0 \in (0,1)$ *and*

$$\mathrm{N}_{\varepsilon_0,m} := \{n \in \mathbb{N}^* : \mu_n(-m) \leq m\sqrt{1-\varepsilon_0}\}\,.$$

There exist C_-, C_+, m_0 *such that, for all* $m \geq m_0$ *and* $n \in \mathrm{N}_{\varepsilon_0,m}$,

$$\mu_n^-(m) \leq \mu_n(-m) \leq \mu_n^+(m)\,,$$

with $\mu_n^{\pm}(m)$*being the n-th eigenvalue of the operators* $\mathcal{L}_m^{\Gamma,\pm}$ *of* $L^2(\Gamma,\mathbb{C})^4$ *defined by*

$$\mathcal{L}_m^{\Gamma,-} = \left([1 - C_- m^{-\frac{1}{2}}]\mathcal{L}^\Gamma - \frac{\kappa^2}{4} + K - C_- m^{-1}\right)_+^{\frac{1}{2}}\,,$$

$$\mathcal{L}_m^{\Gamma,+} = \left([1 + C_+ m^{-\frac{1}{2}}]\mathcal{L}^\Gamma - \frac{\kappa^2}{4} + K + C_+ m^{-1}\right)^{\frac{1}{2}}\,.$$

Corollary

For all $n \in \mathbb{N}^*$, *we have that*

$$\mu_n(-m) \underset{m\to+\infty}{=} \widetilde{\mu}_n^{\frac{1}{2}} + \mathcal{O}(m^{-\frac{1}{2}}),$$

where $(\widetilde{\mu}_n)_{n\in\mathbb{N}^*}$ *is the non-decreasing sequence of the eigenvalues of the following non-negative operator on* $L^2(\Gamma,\mathbb{C})^4 \cap \ker(1_4 - \mathcal{B})$:

$$\mathcal{L}^\Gamma - \frac{\kappa^2}{4} + K\,.$$

When $\Omega = B(0,R),\ R > 0$. Let $A = \beta(1 + 2S\cdot L)$ where $S = \frac{1}{2}\gamma_5\alpha$ and $L = \mathbf{x}\times D$. We have

$$A\mathcal{B} = \mathcal{B}A\,, \qquad \mathcal{L}^\Gamma - \frac{\kappa^2}{4} + K = R^{-2}A^2\,,$$

and its spectrum is $\{n^2/R^2,\ n \in \mathbb{N}^*\}$.

Semiclassical reformulation

Now we rather consider $\left(H^{\Omega}_{-m}\right)^2$ and introduce the semiclassical parameter

$$h = m^{-2} \to 0\,.$$

and the semiclassical operator

$$\mathscr{L}_h = h^2((H^{\Omega}_{-m})^2 - m^2 1_4)\,,$$

whose domain is given by

$$\begin{aligned}\mathrm{Dom}(\mathscr{L}_h) &= \mathrm{Dom}((H^{\Omega}_{-m})^2)\\ &= \Big\{\psi \in H^2(\Omega)\ :\ \psi \in \ker(\mathcal{B} - 1_4)\,,\\ &\qquad \left(\partial_{\mathbf{n}} + \frac{\kappa}{2} - h^{-\frac{1}{2}}\right)\psi \in \ker(\mathcal{B} + 1_4)\,,\ \text{on } \Gamma\Big\}.\end{aligned}$$

The associated quadratic $\mathscr{Q}_h$ form is defined by

$$\forall \psi \in \mathrm{Dom}(\mathscr{Q}_h)\,, \quad \mathscr{Q}_h(\psi) = h^2 \|\nabla \psi\|^2_{L^2(\Omega)} + \int_\Gamma \left(\frac{\kappa}{2} h^2 - h^{\frac{3}{2}}\right) |\psi|^2 d\Gamma\,,$$

where

$$\mathrm{Dom}(\mathscr{Q}_h) = \mathrm{Dom}(H^\Omega_{-m}) = \left\{\psi \in H^1(\Omega) \ : \ \psi \in \ker(\mathcal{B} - 1_4) \text{ on } \Gamma\right\}\,.$$

The operator $\mathscr{L}_h$ is the semiclassical Laplacian with combined MIT bag condition *and* Robin condition on the boundary.

Relations between the eigenvalues of $\mathscr{L}_h$ and H^Ω_{-m}

Recall that the spectrum of H^Ω_{-m} is discrete, symmetric with respect to 0 and with pair multiplicity.
The spectrum of H^Ω_{-m} lying in $[-m, m]$ is given by

$$\left\{ \pm\sqrt{h^{-2}\lambda_n(h) + h^{-1}} \,:\, n \in \mathbb{N}^*, -h \leq \lambda_n(h) \leq 0 \right\},$$

where $\lambda_n(h)$ denotes the n-th eigenvalue of $\mathscr{L}_h$.
Therefore, we shall focus on the study of the negative eigenvalues of $\mathscr{L}_h$.
Remark: The theorem for negative mass shares common features with the known results about the Robin Laplacian in the semiclassical limit. A major difference is that the effective operator is a quadratic function of the principal curvatures while is linear in the Robin case. This is due to the vectorial nature of the MIT operator.

Main steps of the proof

(a) By using an Agmon type estimate we see that the eigenfunctions are localized near the boundary at a scale of order $h^{\frac{1}{2}}$. Hence, we redefine the operator near the boundary.

(b) We rewrite the operator near the boundary in tubular coordinates $(s, t) \in \Gamma \times (0, \delta)$.

(c) We perform a change of scale in the normal direction, $(\sigma, \tau) = (s, h^{-\frac{1}{2}} t)$, that allows us to see something at the limit.

(d) We relate this operator to a family of one dimensional operators for which we have an estimate of eigenvalues.

We follow the ideas on [2] and [3].

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[2] B. Helffer and A. Kachmar. Eigenvalues for the Robin Laplacian in domains with variable curvature. *To appear in Trans. Amer. Math. Soc.*, 2015.

[3] A. Kachmar, P. Keraval, and N. Raymond. Weyl formulae for the Robin Laplacian in the semiclassical limit. *To appear in Confluentes Math. , 2016.*

Thank you.