Nonrelativistic asymptotics of solitary waves in the Dirac equation with Soler-type nonlinearity

joint work with Andrew Comech (Texas A&M University, College Station)

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A nonlinear Dirac equation

We consider the spectral stability of stationary solutions $\phi_\omega(x)e^{-i\omega t}$ to a nonlinear Dirac equation of the form

$$\mathrm{i}\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi, \qquad \psi(x,t)\in\mathbb{C}^N, \quad x\in\mathbb{R}^n, \quad \text{(NLD)}$$

where N is even, $f(0)=0$, and D_m is the free Dirac operator:

$$D_m = -\mathrm{i}\alpha\cdot\nabla + \beta m = \sum_{\jmath=1}^{n} -\mathrm{i}\alpha^\jmath\partial_{x_\jmath} + \beta m \qquad m>0.$$

The $N\times N$ Dirac matrices are hermitian and satisfy $1\le \jmath,k\le n$

$$(\alpha^\jmath)^2 = \beta^2 = I_N, \qquad \alpha^\jmath\alpha^k + \alpha^k\alpha^\jmath = 2\delta_{\jmath k}I_N, \qquad \alpha^\jmath\beta + \beta\alpha^\jmath = 0.$$

Its spectrum is purely absolutely continuous and given by

$$\mathbb{R}\setminus(-m,m).$$

We consider values of ω in open interval of $(-m,m)$.

The linearization

We consider the solution to the nonlinear Dirac equation in the form

$$\psi(x,t) = (\phi_\omega(x) + \rho(x,t))e^{-i\omega t},$$

where ϕ_ω satisfies the stationary equation

$$\omega\phi_\omega = D_m\phi_\omega - f(\phi_\omega^*\beta\phi_\omega)\beta\phi_\omega,$$

so that $\rho(x,t) \in \mathbb{C}^N$ is a "small" perturbation of $\phi_\omega(x)e^{-i\omega t}$. The linearization at a solitary wave (the linearized equation on ρ) is given by

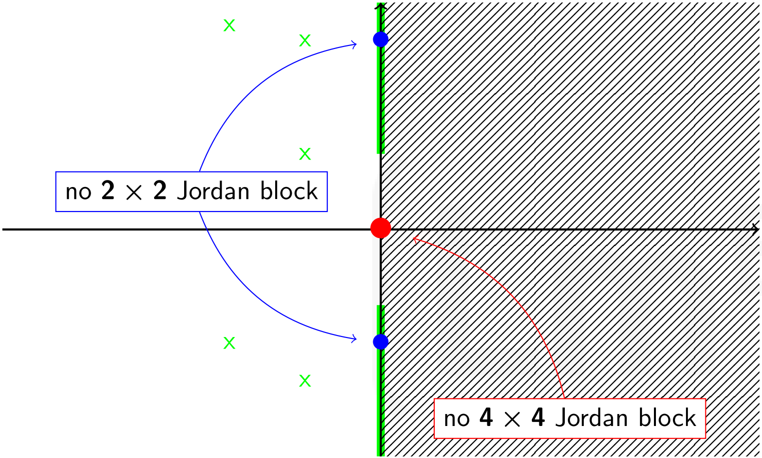
$$\partial_t\rho = \mathrm{J}\mathrm{L}(\omega)\rho,$$

where $\mathrm{J} = 1/i$,

$$\mathrm{L}(\omega) = D_m - \omega - f(\phi_\omega^*\beta\phi_\omega)\beta - 2\Re(\phi_\omega^*\beta\ \cdot)f'(\phi_\omega^*\beta\phi_\omega)\beta\phi_\omega,$$

Definition

A solitary wave is spectrally stable if the spectrum of the corresponding linearization does not contain any point $\boldsymbol{\lambda}$ with positive real part there is not any Jordan block of order larger than $\mathbf{4}$ at $\boldsymbol{\lambda} = \mathbf{0}$ and not any non trivial Jordan bloc at $\boldsymbol{\lambda} \in \mathbf{i}\mathbb{R} \setminus \mathbf{0}$.



Definition

A solitary wave is spectrally stable if the spectrum of the corresponding linearization does not contain any point $\boldsymbol{\lambda}$ with positive real part there is not any Jordan block of order larger than $\mathbf{4}$ at $\boldsymbol{\lambda} = \mathbf{0}$ and not any non trivial Jordan bloc at $\boldsymbol{\lambda} \in \mathbf{i}\mathbb{R} \setminus \mathbf{0}$.

- The essential spectrum of $\mathbf{JL}(\omega)$ is purely imaginary and its thresholds are $\pm(\boldsymbol{m} - |\omega|)\mathbf{i}$ (Weyl's theorem).
- There are embedded thresholds $\pm(\boldsymbol{m} + |\omega|)\mathbf{i}$.

For the spectral stability, only the point spectrum and even the discrete spectrum are relevant.

Notice that

$$\mathbf{Span}\,\{\mathbf{J}\phi_\omega,\ \partial_{x^j}\phi_\omega\} \subset \ker \mathbf{JL}(\omega),$$

$$\mathbf{Span}\,\{\mathbf{J}\phi_\omega,\ \partial_\omega\phi_\omega,\ \partial_{x^j}\phi_\omega,\ \alpha^j\phi_\omega - 2\omega x^j\mathbf{J}\phi_\omega\} \subset \mathcal{N}_g(\mathbf{JL}(\omega)).$$

The nonlinear Schrödinger equation

For the ground state solution $\phi_\omega(x)e^{-i\omega t}$ of the a nonlinear Schrödinger equation

$$\mathbf{i}\partial_t\psi = -\mathbf{\Delta}\psi - |\psi|^{2k}\psi, \qquad \psi(x,t)\in\mathbb{C}, \quad x\in\mathbb{R}^n, \qquad \text{(NLS)}$$

where $k>0$, the linearization is given by

$$\partial_t\rho = \mathrm{jl}(\omega)\rho,$$

where

$$\mathrm{jl}(\omega) := \begin{pmatrix} 0 & \mathrm{l}_-(\omega) \\ -\mathrm{l}_+(\omega) & 0 \end{pmatrix}$$

where $\mathrm{j}\sim 1/\mathbf{i}$,

$$\mathrm{l}_+(\omega) = \mathrm{l}_-(\omega) - 2k\Re(\phi_\omega{}^* \cdot)|\phi_\omega|^{2(k-1)}\phi_\omega \quad \mathrm{l}_-(\omega) = -\mathbf{\Delta} - \omega - |\phi_\omega|^{2k}$$

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For some $c>0$, we have

$$\mathrm{l}_-(\omega)\phi_\omega = 0 \quad \mathrm{l}_-(\omega) > cI_{\phi_\omega^\perp},\ c>0.$$

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where

$$\mathrm{j}\mathrm{l}(\omega) := \begin{pmatrix} 0 & \mathrm{l}_-(\omega) \\ -\mathrm{l}_+(\omega) & 0 \end{pmatrix}$$

$$\mathrm{j}\mathrm{l}(\omega)\rho = \lambda\rho \Rightarrow \mathrm{l}_+(\omega)\mathrm{l}_-(\omega)\rho_2 = -\lambda^2\rho_2$$
$$\Rightarrow \sqrt{\mathrm{l}_-(\omega)}\mathrm{l}_+(\omega)\sqrt{\mathrm{l}_-(\omega)}R = -\lambda^2 R$$

for $R = \sqrt{\mathrm{l}_-(\omega)}\rho_2$ where ρ_2 is the second component of ρ.

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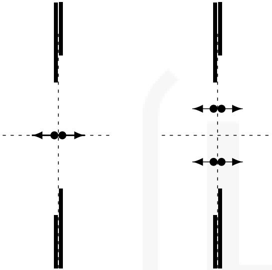
So

$$\sigma(\mathrm{jl}(\omega)) \subset \mathbb{R}\cup i\mathbb{R}.$$

This no longer true for nonlinear Dirac equations.

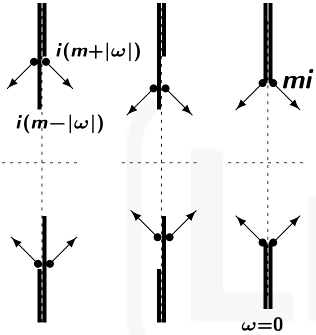
The possible “scenari”

Birth of real eigenvalues out of collisions of eigenvalues



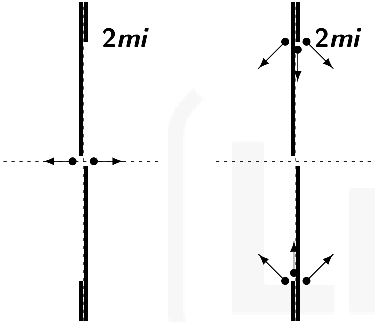
The possible "scenari"

Possible bifurcations from the essential spectrum



The possible "scenari"

Bifurcations from $\boldsymbol{\lambda = 0}$ and hypothetical bifurcations from $\boldsymbol{\lambda = \pm 2mi}$ in the nonrelativistic limit, $\omega \lesssim m$.



Hypothesis

$f \in C(\mathbb{R})$ and there exist $k > 0$, and $c > 0$ such that

$$|f(s) - |s|^k| = o(|s|^k), \qquad s \in \mathbb{R}.$$

If $n \geq 3$ then $k < 2/(n-2)$.

Consider the matrix β in the form:

$$\beta = \pm \begin{bmatrix} I_{N/2} & 0 \\ 0 & -I_{N/2} \end{bmatrix}$$

the matrices $(\alpha^j)_{1 \leq j \leq n}$ are of the form

$$\alpha^j = \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix}, \qquad 1 \leq j \leq n,$$

where the $(\sigma_j)_{1 \leq j \leq n}$ are hermitian and satisfies

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk}, \qquad 1 \leq j, k \leq n.$$

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$$\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}, \qquad 1 \leq j, k \leq n.$$

We consider the existence of solitary waves of Soler or Wakano type $\phi_\omega(x)e^{-i\omega t}$ with

$$\phi_\omega = \begin{bmatrix} v(r) n_1 \\ u(r)\,(e_r \cdot \sigma)\, n_1 \end{bmatrix}$$

if $\varsigma = 1$.
The profiles v and u are real and

$$n_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{N/2}, \qquad e_r = \frac{x}{r} \in \mathbb{R}^n, \qquad \sigma = (\sigma_\jmath)_{1 \le \jmath \le n}.$$

From (NLD), we deduce

$$\begin{cases} \partial_r u + \frac{n-1}{r} u + (m-\omega) v = f(v^2 - u^2) v, \\ \partial_r v + (m+\omega) u = f(v^2 - u^2) u, \end{cases} \qquad r > 0.$$

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$$\phi_\omega = \begin{bmatrix} u(r)\,(e_r\cdot\sigma)\,n_1 \\ v(r)n_1 \end{bmatrix}$$

if $\varsigma = -1$.
The profiles v and u are real and

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Theorem

There exist ω_0 *and, for* $\omega \in (\omega_0, m)$*, a solution of the form:*

$$v(r,\omega) = \epsilon^{\frac{1}{k}} \left[\hat{V}(\epsilon r) + \tilde{V}(\epsilon r, \epsilon)\right], u(r,\omega) = \epsilon^{1+\frac{1}{k}} \left[\hat{U}(\epsilon r) + \tilde{U}(\epsilon r, \epsilon)\right],$$

where ϵ *and* ω *verify* $\epsilon = \sqrt{m^2 - \omega^2}$, $\hat{V}(t) = u_k(|t|)$ *is even, positive, exponentially decreasing and* C^2 *with*

$$-\frac{1}{2m}\hat{V} = -\frac{1}{2m}\left(\partial_t^2 + \frac{n-1}{t}\partial_t\right)\hat{V} - \hat{V}^{2k+1},$$

and $\hat{U}(t) = -\hat{V}'(t)/(2m)$.
There exists $\tau > 0$ *such that* $\tilde{V}$ *and* $\tilde{U}$ *verify*

$$\|e^{\tau\langle r\rangle}\tilde{V}\|_{H^1} + \|e^{\tau\langle r\rangle}\tilde{U}\|_{H^1} = O(1).$$

The equation for the couple $(\tilde{V}, \tilde{U})$ is given by:

$$\begin{cases} (\partial_t + \frac{n-1}{t})\tilde{U} + \frac{1}{m+\omega}\tilde{V} = (1+2k)|\hat{V}|^{2k}\tilde{V} - G_1(\epsilon, \tilde{V}, \tilde{U}), \\ \partial_t \tilde{V} + (m+\omega)\tilde{U} = G_2(\epsilon, \tilde{V}, \tilde{U}), \end{cases}$$

for $t \in \mathbb{R}$, $\epsilon > 0$, where

$$\begin{aligned} G_1(\epsilon, \tilde{V}, \tilde{U}) &= -\epsilon^{-2} f\left(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)\right) V + \hat{V}^{2k}\hat{V} + (1+2k)\hat{V}^{2k}\tilde{V} \\ &\quad + \left(\frac{1}{m+\omega} - \frac{1}{2m}\right)\hat{V}, \end{aligned}$$

$$G_2(\epsilon, \tilde{V}, \tilde{U}) = f\left(\epsilon^{2/k}(V^2 - \epsilon^2 U^2)\right) U + (m-\omega)\hat{U},$$

and $\omega = \sqrt{m^2 - \epsilon^2}$.

Let

$$G(\epsilon, \tilde{W}) = \begin{bmatrix} G_1(\epsilon, \tilde{V}, \tilde{U}) \\ G_2(\epsilon, \tilde{V}, \tilde{U}) \end{bmatrix}, \qquad \tilde{W} = \begin{bmatrix} \tilde{V} \\ \tilde{U} \end{bmatrix},$$

and

$$A(\epsilon) = \begin{bmatrix} -\frac{1}{m+\omega} + (1+2k)|\hat{V}|^{2k} & -\partial_t - \frac{n-1}{t} \\ \partial_t & m+\omega \end{bmatrix}, \qquad \omega = \sqrt{m^2 - \epsilon^2},$$

with domain

$$D(A(\epsilon)) = H^1_{e,o}(\mathbb{R}, |t|^{n-1}\mathrm{d}t;\ \mathbb{C}^2),$$

where

$$H^1_{e,o}(\mathbb{R}, |t|^{n-1}\mathrm{d}t;\ \mathbb{C}^2) := H^1_{\mathrm{even}}(\mathbb{R}, |t|^{n-1}\mathrm{d}t; \mathbb{C}) \times H^1_{\mathrm{odd}}(\mathbb{R}, |t|^{n-1}\mathrm{d}t;\ \mathbb{C}),$$

similarly for $L^2_{e,o}(\mathbb{R}, |t|^{n-1}\mathrm{d}t; \mathbb{C}^2)$, and

$$A(\epsilon):\ H^1_{e,o}(\mathbb{R}, |t|^{n-1}\mathrm{d}t;\ \mathbb{C}^2) \to L^2_{e,o}(\mathbb{R}, |t|^{n-1}\mathrm{d}t;\ \mathbb{C}^2).$$

The system takes the form

$$A(\epsilon)\tilde{W}(t,\epsilon) = G(\epsilon, \tilde{W}(t,\epsilon)), \qquad \epsilon > 0.$$

We have

- $\sigma_{\mathrm{ess}}\big(A(\epsilon)\big) = \Big(-\infty, -\frac{1}{m+\omega}\Big] \cup \Big[m+\omega, +\infty\Big)$, $\epsilon \in [0, m]$.
- $\ker A(0)\big|_{H^1_{e,o}(\mathbb{R},|t|^{n-1}\mathrm{d}t;\,\mathbb{C}^2)} = \{0\}$.

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \ker A(0)$$
$$\Longrightarrow \eta(t) = -\frac{1}{2m}\xi'(t) \text{ for } t \in \mathbb{R}$$
$$\Longrightarrow \xi(|x|) \in \ker \mathrm{l}_+ \text{ for } x \in \mathbb{R}^n,$$

The restriction of l_+ to spherically symmetric functions has zero kernel

$$\Rightarrow \lambda = 0 \notin \sigma\big(A(0)\big|_{L^2_{e,o}}\big)$$

$A(0)^{-1}$ is bounded from $L^2_{e,o}(\mathbb{R}, |t|^{n-1}\mathrm{d}t;\, \mathbb{C}^2)$ to $H^1_{e,o}(\mathbb{R}, |t|^{n-1}\mathrm{d}t;\, \mathbb{C}^2)$.

The solitary waves we are looking for are fixed points of the mapping

$$\mu_\gamma(\epsilon,\cdot):\ L^2_{e,o}(\mathbb{R},|t|^{n-1}\mathrm{d}t;\ \mathbb{C}^2)\cap L^\infty(\mathbb{R};\ \mathbb{C}^2)\to H^1_{e,o}(\mathbb{R},\langle t\rangle^{n-1}\mathrm{d}t;\ \mathbb{C}^2),$$
$$Z\mapsto e^{-2k\gamma\langle t\rangle}A_\gamma(\epsilon)^{-1}e^{(1+2k)\gamma\langle t\rangle}G(\epsilon,e^{-\gamma\langle t\rangle}Z)$$

with

$$A_\gamma(\epsilon):=e^{(1+2k)\gamma\langle t\rangle}\circ A(\epsilon)\circ e^{-(1+2k)\gamma\langle t\rangle}=A(\epsilon)-(1+2k)\gamma\frac{t}{\langle t\rangle}.$$

There is $a_0>0$ such that for $\Lambda_k:=\sup_{x\in\mathbb{R}^n}|\hat{V}(x)|+m\sup_{x\in\mathbb{R}^n}|\hat{U}(x)|$

$$\mu_\gamma\left(\epsilon,\ \overline{\mathbb{B}_\rho(X_{e,o})}\right)\subset\overline{\mathbb{B}_\rho(X^1_{e,o})},\quad \rho=a_0\max\left(H(\epsilon^{2/k}4\Lambda_k^2),\ \epsilon^{2k},\ \epsilon^2\right)$$

with H such that is monotonically increasing with $H(0)=0$ and

$$|f(\tau)-|\tau|^k|\leq|\tau|^kH(\tau).$$

We conclude with Schauder fixed point theorem.

Lemma

If $f \in C(\mathbb{R})$ *and if* V, $U \in C(\mathbb{R})$, *with* V *even and* U *odd, then*

- V, $U \in C^1(\mathbb{R})$
- $U(t)/t$, $t \neq 0$ *could be extended to a continuous function on* $\mathbb{R}$.
- *if there is* $C < \infty$ *such that*

$$|V(t)| + |U(t)| \leq C, \qquad \forall t \in \mathbb{R},$$

then there is $C' < \infty$ *such that*

$$|\partial_t V(t)| + |\partial_t U(t)| \leq C', \qquad \forall t \in \mathbb{R}.$$

Back to the linearization

The linearized equation (in ρ) is given by

$$\mathrm{i}\partial_t \rho = \mathcal{L}(\omega)\rho,$$

with

$$\mathcal{L}(\omega) = D_m - \omega - f(\phi_\omega^* \beta \phi_\omega)\beta - 2f'(\phi_\omega^* \beta \phi_\omega)\beta\phi_\omega \Re(\phi_\omega^* \beta\ \cdot).$$

This term is singular where

$$\phi_\omega^* \beta \phi_\omega$$

vanishes if $f(s) = |s|^k$ for $k \in (0, 1)$.

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Proposition

There exist $\epsilon_1 \in (0, \epsilon_0)$ *such that if* $\epsilon \in (0, \epsilon_1)$ *then*

$$\epsilon |U(t,\epsilon)| < V(t,\epsilon)/2, \qquad t \in \mathbb{R}, \qquad \epsilon \in (0, \epsilon_1);$$

$$\phi_\omega^*(x)\beta\phi_\omega(x) = |V(|x|)|^2 - |U(|x|)|^2 \geq \frac{1}{2}(|V(|x|)|^2 + |U(|x|)|^2).$$

$$U = \hat{U} + \tilde{U} \quad V = \hat{V} + \tilde{V}.$$

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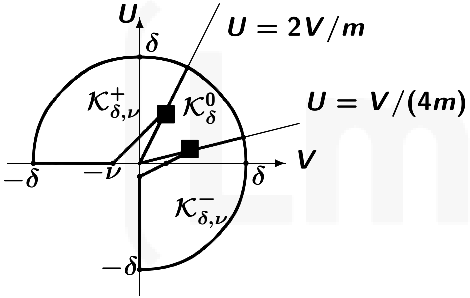

Hypothesis

$f \in C(\mathbb{R})$ and there exist k, $K \in \mathbb{R}$, $K > k > 0$, and $c > 0$ such that

$$|f(s) - |s|^k| \leq c|s|^K, \qquad s \in \mathbb{R}.$$

If $n \geq 3$ then $k < 2/(n-2)$.

We have the improved estimates

$$\|e^{\tau\langle r\rangle}\tilde{V}\|_{H^1} + \|e^{\tau\langle r\rangle}\tilde{U}\|_{H^1} = O(\epsilon^{2\varkappa}),$$

where $\varkappa = \min\left(1, \frac{K}{k} - 1\right) > 0$.

Hypothesis

$f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$ *and that there are* $k > 0$ *and* $K > k$ *such that*

$$|f(\tau) - |\tau|^k| = O(|\tau|^K), \qquad |\tau| \le 1;$$
$$|\tau f'(\tau) - k|\tau|^k| = O(|\tau|^K), \qquad |\tau| \le 1.$$

Theorem

There is ϵ_2 *small enough so that for* $\omega = \sqrt{m^2 - \epsilon^2}$, $\epsilon \in (0, \epsilon_2)$, *the functions* $\phi_\omega(x)$, $\tilde{V}(t,\epsilon)$, *and* $\tilde{U}(t,\epsilon)$ *are unique.*

Moreover, the map $\omega \mapsto \phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N)$, *is* C^1 *and*

$$\|e^{\gamma\langle t\rangle}\partial_\epsilon \begin{bmatrix} \tilde{V}(t,\epsilon) \\ \tilde{U}(t,\epsilon) \end{bmatrix}\|_{H^1(\mathbb{R},\mathbb{R}^2)} = O(\epsilon^{2\varkappa-1}), \qquad \epsilon \in (0, \epsilon_0),$$

and there is $b > 0$ *such that*

$$\|\partial_\omega \phi_\omega\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} = b\epsilon^{-n+\frac{2}{k}}(1 + O(\epsilon^{2\varkappa})).$$

Additionally, assume that either $\boldsymbol{k} < \mathbf{2}/\boldsymbol{n}$, or $\boldsymbol{k} = \mathbf{2}/\boldsymbol{n}$ and $\boldsymbol{K} > \mathbf{4}/\boldsymbol{n}$. Then there is $\omega_{\mathbf{1}} < \boldsymbol{m}$ such that $\partial_\omega \boldsymbol{Q}(\omega) < \mathbf{0}$ for all $\omega \in (\omega_{\mathbf{1}}, \boldsymbol{m})$.

If instead $\boldsymbol{k} > \mathbf{2}/\boldsymbol{n}$, then there is $\omega_{\mathbf{1}} < \boldsymbol{m}$ such that $\partial_\omega \boldsymbol{Q}(\omega) > \mathbf{0}$ for all $\omega \in (\omega_{\mathbf{1}}, \boldsymbol{m})$.