On the dispersive dynamics of the Dirac equation with critical potentials

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Dispersive estimates for the massless Dirac equation

The solution to the Cauchy problem

$$\begin{cases} i\partial_t u + \mathcal{D}u = 0, & u(t,x) : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}^4 \\ u(0,x) = f(x) \end{cases} \tag{1}$$

where $\mathcal{D} = i^{-1} \sum_{j=1}^{3} \alpha_j \partial_j$ is the 3D Dirac operator satisfies different sets of *dispersive estimates*.

- *Strichartz estimates:*

$$\| e^{it\mathcal{D}} f \|_{L^p_t \dot{H}_x^{\frac{1}{q} - \frac{1}{p} - \frac{1}{2}}} \lesssim \|f\|_{L^2}, \tag{2}$$

$$\frac{2}{p} + \frac{2}{q} = 1, \quad 2 < p \leq \infty, \quad 2 \leq q < \infty.$$

Applications: crucial tool in the study of nonlinear problems: local/global well posedness, scattering...

Question: do Strichartz estimates hold in presence of perturbations?

Local smoothing estimates

We consider the dynamics perturbed with an electric potential

$$\begin{cases} i\partial_t u + \mathcal{D}u + Vu = 0, \\ u(0,x) = f(x). \end{cases} \tag{3}$$

We consider in particular potentials in the form $V(x) = \frac{1}{|x|^a}$ with $a > 0$.

The general picture. The homogeneity of the operator works as a threshold for the validity of Strichartz estimates. In other words, if we consider the behaviour at $|x| >> 1$, if $a > 1$ then Strichartz estimates hold; if $a < 1$ then one expects to find counterexamples.

The case $a = 1$ (Coulomb potential) then represents the threshold: it is the scaling critical case. Philosophically, it can be compared to the Schrödinger equation with inverse square potential.

Intermediate estimates: local smoothing estimates take the form

$$\|w(x)^{-1} e^{it(\mathcal{D}+V)} f\|_{L^2_t L^2_x} \leq \|f\|_{L^2} \tag{4}$$

for some weight function $w(x)$.

From local smoothing to Strichartz (a relevant example)

Suppose we can prove for the Dirac equation with a Coulomb potential the following estimate for $u = e^{it(\mathcal{D}+\frac{\nu}{|x|})}f$

$$\||x|^{-1/2}u\|_{L^2_t L^2_x} \leq \|f\|_{L^2}, \tag{5}$$

then denoting by $S_{p,q}$ any admissible Strichartz space we have

$$\|u\|_{S_{p,q}} \lesssim \|e^{it\mathcal{D}}f\|_{S_{p,q}} + \left\| e^{it\mathcal{D}} \int_0^t e^{-is\mathcal{D}} \left(\frac{\nu}{|x|} u(s) \right) ds \right\|_{S_{p,q}}. \tag{6}$$

The perturbation term can be estimated again using free Strichartz and the dual of local smoothing estimate, i.e.

$$\left\| \int e^{-is(\mathcal{D}+\frac{\nu}{|x|})} F(s) ds \right\|_{L^2} \leq \||x|^{1/2}F\|_{L^2_t L^2_x}$$

in the case $\nu = 0$, thus yielding

$$(6) \lesssim \left\| \int_0^t e^{-is\mathcal{D}} \left(\frac{\nu}{|x|} u(s) \right) ds \right\|_{L^2} \leq \nu \left\| |x|^{-1/2} u \right\|_{L^2_t L^2_x} \lesssim \|f\|_{L^2}$$

Schrödinger equation with inverse square: our inspiration

In particular, in 2003 Burq-Planchon-Stalker-Tahvlidar Zadeh proved the following local smoothing estimate

$$\| |x|^{-1/2-2\alpha} (P_a^{1/4-\alpha}) u \|_{L^2_t L^2_x} \leq C \|f\|_{L^2(\mathbb{R}^n)}$$

for $n \geq 2$, $\alpha \in (0, \frac{1}{4} + \frac{1}{2}\mu_d)$, $\mu_d = \sqrt{(\lambda(n)+d)^2 + a}$, $d \geq 0$, $P_a = -\Delta + \frac{a}{|x|^2}$, $\lambda(n) = \frac{n-2}{2}$ and u solves

$$\begin{cases} i\partial_t u - P_a u = 0, \\ u(0,x) = f(x) \in L^2_{\geq d(\mathbb{R}^n)} \end{cases}$$

where with $L^2_{\geq d}$ we are denoting the subspace of L^2 consisting of all functions that are orthogonal to all spherical harmonics of degree less than d.

The proof of Burq,Planchon,Stalker,Tahvildar-Zadeh

Aim: proving a local smoothing estimate of the form

$$\||x|^{-1}u\|_{L^2(\mathbb{R}^{n+1})} \leq C\|f\|_{L^2(\mathbb{R}^n)}$$

for equation

$$\begin{cases} i\partial_t u + \Delta u - \dfrac{a}{|x|^2}u = 0, \\ u(0,x) = f(x) \end{cases} \qquad (7)$$

- Use spherical harmonics decomposition $f = \sum_{l,m} f_l^m(r) Y_l^m(\theta)$, rotational symmetry and L^2 orthogonality of spherical harmonics to reduceto a radial problem on every (fixed) l-th space. On the l-th spherical space the operator $-\Delta + \frac{a}{|x|^2}$ becomes

$$A_\mu = -\partial_r^2 - (n-1)r^{-1}\partial_r + [l(l+n-2)+a]r^{-2};$$

- Use Hankel transform

$$(\mathcal{H}_\mu u)(\xi) = \int_0^{+\infty} (r|\xi|)^{\frac{2-n}{2}} J_\mu(r|\xi|) u(r\xi/|\xi|) r^{n-1}$$

which has the properties:

1. $\mathcal{H}_\mu^2 = \mathrm{Id}$;
2. $\mathcal{H}_\mu$ is an L^2-isometry;
3. $\mathcal{H}_\mu A_\mu = \Omega^2 \mathcal{H}_\mu$, where the operator $\Omega^a f(x) = |x|^a f(x)$.
4. Using (3) define fractional powers of A_μ as

$$A_\mu^{\sigma/2} u(r,\theta) = \mathcal{H}_\mu \Omega^\sigma \mathcal{H}_\mu u(r,\theta) = \int_0^\infty k_\mu^\sigma(r,s) u(s,\theta) s^{n-1} ds$$

with a certain integral kernel k (which is explicitly known).

Now the proof: by applying Hankel transform we can rewrite

$$\| |x|^{-1} u \|_{L^2_t L^2_x} = \| A_\mu^{-1/2} \mathcal{H}_\mu u \|_{L^2_t L^2_x}. \tag{8}$$

On the other hand, $\mathcal{H}_\mu u$ solves

$$i\partial_t \mathcal{H}_\mu u - \Omega^2 \mathcal{H}_\mu u = 0, \qquad \mathcal{H}_\mu u(\xi, 0) = \mathcal{H}_\mu f(\xi)$$

and thus takes the form, after Fourier transforming in time, $\mathcal{F}_t \mathcal{H}_\mu u(\tau, \xi) = \mathcal{H}_\mu f(\xi) \delta(\tau - |\xi|^2)$. The L^2 norm in (8) then becomes,

$$\| A_\mu^{-1/2} \mathcal{F}_t \mathcal{H}_\mu u \|_{L^2 L^2} = \left\| \int_0^\infty k_\mu^{-1}(|\xi|, s) \delta(\tau - s^2) \mathcal{H}_\mu f(s\xi/|\xi|) s^{n-1} ds \right\|_{L^2 L^2}$$

$$\cong \int_0^\infty \int_{S^{n-1}} \tau^{4\lambda(n)+2} k_\mu^{-2}(\tau, \tau) |(\mathcal{H}_\mu f)(\tau\theta)|^2 d\theta = C_\mu \| \mathcal{H}_\mu f \|_{L^2}.$$

Crucial: The diagonal values of the integral kernel are

$$k_\mu^{-2}(\tau, \tau) \cong C_\mu \tau^{n-1-(4\lambda(n)+2)}.$$

The massless Dirac-Coulomb equation

We consider the problem

$$\begin{cases} i\partial_t u + \mathcal{D}u + \frac{\nu}{|x|}u = 0, \quad u(t,x) : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}^4 \\ u(0,x) = f(x) \end{cases} \tag{9}$$

with $\nu \in (-1,1)$. We aim for an estimate of the form

$$\left\| |x|^{-a} |\mathcal{D}_\nu|^b e^{it(\mathcal{D}+\frac{\nu}{|x|})} f \right\|_{L^2_t L^2_x} \leq C \|f\|_{L^2}.$$

with $\mathcal{D}_\nu = \mathcal{D} + \frac{\nu}{|x|}$.

Problems:

- We are dealing with a system;
- The "spherical harmonic" decomposition is more subtle;
- Building an explicit Hankel transform is tricky;
- The corresponding integral kernel is difficult to handle (the explicit form of generalized eigenstates involves special functions)

The result

Theorem (F.C., E. Seré, JFA '16)

Let $\nu \in (-1, 1)$, *u be solution of* (9) *and let* $j \in \frac{1}{2} + \mathbb{N}$. *Then for any*

$$1/2 < \alpha < \sqrt{(j + 1/2)^2 - \nu^2} + 1/2$$

and any $f \in L^2((0, \infty)) \otimes \mathcal{H}_{\geq j}$ *there exists a constant* $C = C(\nu, \alpha)$ *such that the following estimate holds*

$$\| |x|^{-\alpha} |\mathcal{D}_\nu|^{1/2-\alpha} u \|_{L^2_t L^2_x} \leq C \|f\|_{L^2}. \tag{10}$$

The setup: partial wave decomposition

Using partial wave subspaces it is possible to decompose every $u \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ as

$$u(x) = \sum_{j,m_j,k_j} f_{m_j,k_j}(r)\Omega^+_{m_j,k_j}(\theta,\phi) + g_{m_j,k_j}(r)\Omega^-_{m_j,k_j}(\theta,\phi) \tag{11}$$

with $j = \frac{1}{2}, \frac{3}{2}, \dots$, $m_j = -j, -j+1, \dots, +j$, $k_j = \pm(j+1/2)$,

$$\Omega^+_{m_j,\mp(j+1/2)} = \frac{i}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m_j}\; Y^{m_j-1/2}_{j-1/2} \\ \sqrt{j-m_j}\; Y^{m_j+1/2}_{j-1/2} \\ 0 \\ 0 \end{pmatrix}$$

$$\Omega^-_{m_j,\mp(j+1/2)} = \frac{1}{\sqrt{2j+2}} \begin{pmatrix} 0 \\ 0 \\ \sqrt{j+1-m_j}\; Y^{m_j-1/2}_{j+1/2} \\ -\sqrt{j+1-m_j}\; Y^{m_j+1/2}_{j+1/2} \end{pmatrix}$$

This decomposition defines a unitary isomorphism between Hilbert spaces

$$L^2(\mathbb{R}^3)^4 \cong \bigoplus_j L^2((0,\infty), dr) \otimes \mathcal{H}_{j,m_j,k_j}.$$

Moreover, the Dirac-Coulomb operator is unitary equivalent to the direct sum of the "partial wave" Dirac operators $\mathcal{D}_{m_j,k_j}$, the action of which with respect to the basis $\{\Omega^+_{m_j,k_j}, \Omega^-_{m_j,k_j}\}$ is given by the radial matrix

$$\mathcal{D}_{k_j} = \begin{pmatrix} \frac{\nu}{r} & -\frac{d}{dr} + \frac{k_j}{r} \\ \frac{d}{dr} + \frac{k_j}{r} & \frac{\nu}{r} \end{pmatrix} \tag{12}$$

which is known as the *radial Dirac operator*.

In what follows, we will thus forget the angular dependance and work on the generical k-th subspace.

The continuous spectrum

The continuous spectrum for the Dirac-Coulomb is well known, and for $\epsilon > 0$ with respect to representation (11) has the radial coordinates

$$\Psi_\epsilon^{k,\nu}(r) = \begin{pmatrix} f_+^{k,\nu}(r,\epsilon) \\ g_+^{k,\nu}(r,\epsilon) \end{pmatrix} = \frac{\pm 2}{\sqrt{\pi}} e^{\frac{1}{2}\pi\nu} \frac{|\Gamma(\gamma+1+i\nu)|}{\Gamma(2\gamma+1)} (\epsilon r)^{\gamma-1} e^{i\epsilon r} \qquad (13)$$

$$\times \left[e^{i\xi} {}_1F_1(\gamma - i\nu, 2\gamma+1, -2i\epsilon r) \mp e^{-i\xi} {}_1F_1(\gamma+1-i\nu, 2\gamma+1, -2i\epsilon r) \right]$$

where $\gamma = \sqrt{k^2 - \nu^2}$ and $e^{-2i\xi} = \frac{\gamma - i\nu}{k}$.

Remark: Using a charge conjugation argument (or working directly on the radial equations) it can be shown that

$$\begin{pmatrix} f_-^{k,\nu}(r,-\epsilon) \\ g_-^{k,\nu}(r,-\epsilon) \end{pmatrix} = \begin{pmatrix} g_+^{-k,-\nu}(r,\epsilon) \\ f_+^{-k,-\nu}(r,\epsilon) \end{pmatrix}.$$

Confluent hypergeometric functions

Confluent hypergeometric functions are given by series of the form

$$ {}_1F_1(a,b,z) = \sum_{l=0}^{\infty} \frac{(a)_l}{(b)_l l!} z^l $$

which are defined for $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus -\mathbb{N}$ and converge for every $z \in \mathbb{C}$ ($(a)_l = a(a+1)\ldots(a+l-1)$ are the Pochhammer symbols).
They are solutions to the differential equation

$$ z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0. $$

The relativistic "Hankel transform"

We introduce the following operator

$$\mathcal{P}_k u(\epsilon) = \begin{pmatrix} \mathcal{P}_k^+ u(\epsilon) \\ \mathcal{P}_k^- u(\epsilon) \end{pmatrix} = \begin{pmatrix} \int_0^{+\infty} \Psi_\epsilon^{k,\nu}(r) u(r) r^2 dr \\ \mathcal{C}\left(\int_0^{+\infty} \Psi_{-\epsilon}^{k,\nu}(r) u(r) r^2 dr\right) \end{pmatrix} \tag{14}$$

$$= \int_0^{+\infty} H_{k,\nu}(\epsilon r) \cdot u(r) r^2 dr$$

where the matrix $H_{k,\nu} = \begin{pmatrix} f_+^{k,\nu}(\epsilon r) & g_+^{k,\nu}(\epsilon r) \\ f_-^{k,\nu}(\epsilon r) & g_-^{k,\nu}(\epsilon r) \end{pmatrix}$.

The operator $\mathcal{P}_k$ plays exactly the role of the Hankel transform for the inverse square Schrödinger equation:

1. $\mathcal{P}_k$ is an L^2-isometry;
2. $\mathcal{P}_k \mathcal{D}_k u = \Omega \mathcal{P}_k u$.
3. The inverse of $\mathcal{P}_k$ is given by $\mathcal{P}_k^{-1} u(r) = \int_0^{+\infty} H_{k,\nu}^*(\epsilon r) \cdot u(\epsilon) \epsilon^2 d\epsilon$.

   where $H_{k,\nu}^* = \begin{pmatrix} f_+^{k,\nu}(\epsilon r) & f_-^{k,\nu}(\epsilon r) \\ g_+^{k,\nu}(\epsilon r) & g_-^{k,\nu}(\epsilon r). \end{pmatrix}$

The crucial interaction operator A_k

Inspired by property (2) we define the following family of operators A_k as

$$A_k^\alpha u(r) = \mathcal{H}_k \Omega^\alpha \mathcal{H}_k^{-1} u_k(r)$$

$$= \int_0^{+\infty} \left(\int_0^{+\infty} H_{k,\nu}(\epsilon r) \cdot H_{k,\nu}^*(\epsilon s) \epsilon^{2+\alpha} d\epsilon \right) u_k(s) s^2 ds.$$

We denote with $S_k(r,s)^\alpha = \int_0^{+\infty} H_{k,\nu}(\epsilon r) \cdot H_{k,\nu}^*(\epsilon s) \epsilon^{2+\alpha}$ the integral kernel of the operator A_k^α.

The proof

We now follow the argument of Burq-Planchon-Stalker-Tahvildar Zadeh:

- Use partial wave subspaces to decompose the problem on the single spaces;
- Use modified Hankel and time Fourier transform to diagonalize the problem;
- Rely on the interaction operator A_k^α to write

$$\| |x|^{-\alpha} |\mathcal{D}_\nu|^{1/2-\alpha} u \|_{L^2(\mathbb{R}^{3+1})}$$

$$\int_0^{+\infty}\int_0^{+\infty} ((\mathcal{H}_k f)^*(\tau) S_k^{-\alpha}(\rho,\tau)^T)\cdot(S_k^{-\alpha}(\rho,\tau)(\mathcal{H}_k f)(\tau))\tau^{5-2\alpha}\rho^2 d\rho d\tau.$$

$$\leq \int_0^{+\infty} \mathrm{Tr}(S_k^{-2\alpha}(\tau,\tau))|\mathcal{H}_k f(\tau)|^2 \tau^{5-2\alpha}\, d\tau.$$

- Crucial step: proving a good estimate on $\mathrm{Tr}(S_k^{-2\alpha}(\tau,\tau))$.
- Apply triangular inequality on partial wave decomposition to conclude the proof

The integral

The crucial step consists thus in the study of the kernel $S_k^{-2\alpha}(r,s)$ which in turns leads to the analysis of integrals of the form

$$I(r,s) =$$

$$C\int_0^{+\infty} \epsilon^{2\gamma-2\alpha} e^{i(r+s)} {}_1F_1(\gamma - i\nu, 2\gamma+1, -2i\epsilon r) {}_1F_1(\gamma - i\nu, 2\gamma+1, -2\epsilon s) d\epsilon \tag{15}$$

in the limit $r \to s$ where the constant

$$C = C_{\gamma,\nu,r,s} = \frac{2}{\pm\pi} e^{\pi\nu} \frac{|\Gamma(\gamma+1+i\nu|^2}{\Gamma(2\gamma+1)^2} (rs)^{\gamma-1}. \tag{16}$$

Proposition

$C_{\gamma,\nu,r,s}\mathcal{R}(I(\tau,\tau)) = c_{\gamma,\nu}\tau^{2\alpha-3}$, *with the constant c bounded in* γ.

The Dirac equation in an Aharonov-Bohm field

We consider the 2D system

$$\begin{cases} iu_t + \mathcal{D}_A u = 0, \quad u(t,x) : \mathbb{R}_t \times \mathbb{R}^2_x \to \mathbb{C}^2 \\ u(0,x) = f(x). \end{cases} \tag{17}$$

where $\mathcal{D}_A = -i \sum_{k=1}^{2} \sigma_k(\partial_k - iA^k)$, with $A(x) = \alpha \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)$.
We adapt the strategy presented above to obtain the following

The result

Theorem (F. C., L. Fanelli)

Let $\alpha \in (0,1)$, *u be a solution of* (17) *and let* $j \in \mathbb{Z}$. *Then for any*

$$1/2 < \gamma < 1 + |l + \alpha|. \tag{18}$$

and any $f \in L^2((0,\infty)rdr) \otimes \mathcal{H}_{\geq l}$ *there exists a constant* $c = c(\alpha, \gamma, l)$ *such that the following estimate holds*

$$\left\| |x|^{-\gamma} \mathcal{D}_A^{1/2-\gamma} u \right\|_{L^2_t L^2_x} \leq c \|f\|_{L^2}. \tag{19}$$

In addition, in the endpoint case $\gamma = 1/2$ *the following estimate holds*

$$\sup_{R>0} R^{-1/2} \|e^{it\mathcal{D}_A} f\|_{L^2_t L^2_{|x|\leq R}} \lesssim \|f\|_{L^2}, \tag{20}$$

The strategy

- Partial wave decomposition (the $2D$ case is slightly easier): we can write indeed

$$\Phi(x) = \sum_{l \in \mathbb{Z}} \frac{1}{2\sqrt{\pi}} \begin{pmatrix} f_l(r) e^{il\phi} \\ i g_l(r) e^{i(l+1)\phi} \end{pmatrix} \qquad (21)$$

- Define the "Hankel transform" as a proper projection onto the spectrum, which is explicit in this case too: we can write indeed generalized eigenstates with respect to decomposition (21) for a fixed value of $l \in \mathbb{Z}$ and energy $E > 0$ as

$$\chi_{l,E}(r) = \begin{pmatrix} f_{l,E}(r) \\ g_{l,E}(r) \end{pmatrix} = \sqrt{\frac{\pi}{2}} \begin{pmatrix} J_{|l+\alpha|}(Er) \\ J_{|l+1+\alpha|}(Er) \end{pmatrix}. \qquad (22)$$

- Interaction integrals (notice that this time we really have Bessel functions: we can use standard Hankel transform).

Comments and problems

- The two results can be combined together, giving local smoothing for the Dirac-Coulomb-AB model.
- This very same technique can be used to prove local smoothing for fractional Schrödinger operators in Aharonov-Bohm magnetic field (joint work with L. Fanelli).
- Estimate (20) requires, after using partial wave decomposition, to prove a bound as

$$\frac{1}{R}\int_0^R \chi_l(r)^2 r dr < C \quad (23)$$

uniform in R and l with $\chi_l(r)$ being the (radial components of the) generalized eigenstate of the perturbed Dirac operator. In the case of $\chi_l(r)$ Bessel functions (free case or AB field), this has been proved by Strichartz. Question: *What for confluent hypergeometric (Dirac-Coulomb)?*
One is tempted to prove

$$\sup_{r,\lambda} |\sqrt{r}\chi_\lambda(r)| \le C$$

-FALSE- even for Bessel. More refined estimates are needed to prove (23)..
- What about the mass?
- What about Strichartz?

Thanks for your attention