Lieb-Thirring type bounds for Dirac and fractional Schrödinger operators with complex potentials

Jean-Claude Cuenin

Ludwig-Maximilians Universität München

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Outline

Statement of the problem

Consider $H = T(p) + V(x)$ in $L^2(\mathbb{R}^d)$ with V complex-valued.

Kinetic energies

- $T(p) = (m^2 + |p|^2)^{s/2} - m^s$
- $T(p) = \sum_{j=1}^d \alpha_j p_j + m\beta$
- $T(p) = \sum_{j=1}^d (1 - \cos(p_j))$
- $\ldots$

Questions

- Where are the discrete (or embedded) eigenvalues of H located?
- Can the eigenvalues be controlled by an L^p norm of V?
- What is the rate of accumulation to $\sigma_{\mathrm{ess}}(H)$?

Lieb-Thirring inequalities (in the self-adjoint case)

Consider $H = -\Delta + V$ with V real-valued. Then

$$\sum_{\lambda \in \sigma_{\mathrm{d}}(H)} |\lambda|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-(x)^{d/2+\gamma} \, \mathrm{d}\, x$$

holds for

$$\begin{aligned} d = 1: &\quad \gamma \geq 1/2 \\ d = 2: &\quad \gamma > 0 \\ d \geq 3: &\quad \gamma \geq 0 \end{aligned}$$

- Lieb-Thirring (1976): $d = 3$, $\gamma = 1$ (stability of matter!)
- Cwikel-Lieb-Rozenblum: $d \geq 3$, $\gamma = 0$ (number of eigenvalues!)
- Semiclassical interpretation
- Sharp constants $L_{\gamma,d}$ known for $d \geq 1$, $\gamma \geq 3/2$ and $d = 1$, $\gamma = 1/2$

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Non-selfadjoint Schrödinger operators (Single eigenvalues)

Consider $H = -\Delta + V$ with V complex-valued. Then

$$\forall z \in \sigma_{\mathrm{d}}(H): \quad |z|^\gamma \leq C_{d,\gamma} \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma} \, \mathrm{d}\, x$$

where

$$\begin{aligned} d = 1: &\quad \gamma = 1/2 \\ d = 2: &\quad 0 < \gamma \leq 1/2 \\ d \geq 3: &\quad 0 \leq \gamma \leq 1/2 \end{aligned}$$

- Abramov, Aslanyan, Davies (2001): $d = 1$ ($C_{1,1/2} = 1/2$)
- Frank (2011): $d \geq 2$ (used uniform resolvent estimates of Kenig, Ruiz, Sogge 1987)
- For radial potentials, Frank and Simon (2015) proved the case $1/2 \leq \gamma \leq d/2$
- In the non-radial case this is still open (Laptev-Safronov conjecture)

Non-selfadjoint Schrödinger operators (eigenvalue sums)

Theorem (Frank, Sabin 2014)

For d, γ as before, $q = d/2 + \gamma$ (and $\gamma \neq 0$) we have

$$\sum_{z \in \sigma_{\mathrm{d}}(H)} \frac{\operatorname{dist}(z, [0, \infty))}{|z|^{(1-\epsilon)/2}} \leq C_{d,q,\epsilon} \, \|V\|_{L^q(\mathbb{R}^d)}^{(1+\epsilon)q/(2q-d)},$$

where

$$\begin{cases} \epsilon > 1 & \text{if } d = 1, \\ \epsilon \geq 0 & \text{if } d \geq 2 \text{ and } d/2 < q < d^2/(2d-1), \\ \epsilon > \frac{(2d-1)q - d^2}{d-q} & \text{if } d \geq 2 \text{ and } d^2/(2d-1) \leq q \leq (d+1)/2. \end{cases}$$

- Improves previous results by Frank, Laptev, Lieb, Seiringer (2006), Demuth, Hansmann, Katriel (2009) and Laptev, Safronov (2009).
- It is not clear what the 'correct' weight should be.

Outline

Birman-Schwinger operator

How to define $H_0 + V$ as a closed operator? We follow Kato, Konno, Kuroda, etc. Assume $V = BA$, where $A : \mathcal{H} \to \mathcal{K}$ and $B : \mathcal{K} \to \mathcal{H}$ are closed densely defined operators such that

a) $AR_0(z) \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $R_0(z)B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

b) $Q(z) := \overline{AR_0(z)B} \in \mathcal{B}(\mathcal{K})$,

c) $-1 \in \rho(Q(z_0))$ for some $z_0 \in \rho(H_0)$.

Then there exists a closed densely defined extension H of $H_0 + V$ whose resolvent is given by

$$R(z) = R_0(z) - \overline{R_0(z)B}\,(I_{\mathcal{K}} + Q(z))^{-1} AR_0(z).$$

d) If $Q(z)$ is compact, then $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0)$ and

$$\dim\ker(H - z) = \dim\ker(I_{\mathcal{K}} + Q(z)) < \infty.$$

Fredholm determinants

Compactness is not enough, we need $Q(z) \in \mathfrak{S}^\alpha(\mathcal{H})$ for some $1 \leq \alpha < \infty$. For all $z \in \rho(H_0)$ we have $m(z, H) = m(z, h)$ where

$$\rho(H_0) \ni z \mapsto h(z) = \mathrm{Det}_{\lceil \alpha \rceil}(I + Q(z)) \in \mathbb{C}.$$

Crucial estimate:

$$\log |h(z)| \leq \Gamma_\alpha \|Q(z)\|_{\mathfrak{S}_\alpha}^\alpha.$$

After conformally maping $\phi : \rho(H_0) \to \mathbb{D}$, this leads to Jensen-type estimates. E.g. if $\|Q(z)\|_{\mathfrak{S}_\alpha}^\alpha \leq C$, then

$$\sum_{z:\, h(z)=0} (1 - |\phi(z)|) < \infty.$$

This is too much to ask. There are generalizations by Borichev, Golinskii and Kupin (2009).

Uniform resolvent estimates for general kinetic energies

Let $T:\mathbb{R}^d\to\mathbb{R}$ be smooth function and fix a regular value λ_0. Take a bump function χ that localizes to a small neighborhood of $T^{-1}(\lambda_0)$.

Theorem (Uniform resolvent estimates in Schatten spaces)

Assume that $S_{\lambda_0}=\{\xi\in\Gamma^*: T(\xi)=\lambda_0\}$ has at least k non-vanishing principal curvatures at every point. Then, locally uniformly for z near λ_0 and for $1\leq q\leq (k+2)/2$,

$$\|A(T(p)-z)^{-1}\chi(p)B\|_{\mathfrak{S}^\alpha(L^2(X))}\leq C\|A\|_{L^{2q}(X)}\|B\|_{L^{2q}(X)},$$

$$\alpha:=\begin{cases}(2d-k-2)q/(d-q) & \text{if } 2d/(2d-k)\leq q\leq (k+2)/2\\ (2d-k-2)q/(d-q)+0 & \text{if } 1\leq q< 2d/(2d-k)\end{cases}$$

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Dirac operator ($d = 1$)

$$H = \begin{pmatrix} mc^2 & -\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}\,x} \\ -\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}\,x} & -mc^2 \end{pmatrix} + V, \quad V \text{ non-Hermitian}$$

Theorem (Cuenin, Laptev, Tretter 2013)

Assume that $\|V\|_1 < 1$. *Then*

$$\sigma_{\mathrm{d}}(H) \subset \overline{B_{mr}(mx_0)} \cup \overline{B_{mr}(-mx_0)},$$

$$x_0 := \sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1 - \|V\|_1^2)} + \frac{1}{2}}, \quad r := \sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1 - \|V\|_1^2)} - \frac{1}{2}}.$$

If $m = 0$, *then* $\sigma_{\mathrm{d}}(H) = \emptyset$.

- Similar bounds on the half-line
- For $H_0 = (m^2 + p^2)^{1/2} - m^2$ no bounds in terms of L^1 possible

Dirac operator ($d \geq 2$)

Theorem (Cuenin 2016)

Let $d \geq 2$, $H_0 = \sum_{j=1}^{d} \alpha_j p_j + \beta$, $V \in L^{(d+1)/2}(\mathbb{R}^d, \mathbb{C}^N) \cap L^d(\mathbb{R}^d, \mathbb{C}^N)$. *Then, for any* $\tau > 0$,

$$\sum_{z \in \sigma_{\mathrm{d}}(H)} \mathrm{dist}(z, \sigma(H_0)) |z^2 - 1|^{\frac{d-1}{2} + \tau} (1 + |z|)^{-2d+1-\tau} \leq C(V, \tau).$$

- Similar estimates for the massless Dirac operator and for fractional Schrödinger operators of order $s < 2d/(d+1)$
- Gives bounds on the number of eigenvalues in certain (bounded) regions

Fractional Schrödinger operator ($d \geq 2$)

Theorem (Cuenin 2016)

Let $d \geq 2$, $H_0 = (-\Delta)^{s/2}$, $s \geq 2d/(d+1)$, *and let* $V \in L^q(\mathbb{R}^d)$, *with* $d/s < q \leq (d+1)/2$. *Then*

$$\sum_{z \in \sigma_{\mathrm{d}}(H_0+V)} |z|^{-(1-\delta)/2} \mathrm{dist}(z, \sigma(H_0)) \leq C_{q,d,s,\delta} \|V\|_{L^q}^{\frac{(1+\delta)sq}{2(sq-d)}},$$

where $\delta \ldots$

- For $s = 2$ this is the bound of Frank and Sabin
- The condition on s arises from an incompatibility between restriction type estimates and Sobolev embedding

Eigenvalue estimates on the lattice

Theorem

Let $T:\mathbb{T}^d\to\mathbb{R}$ *be a smooth function such that for* λ_0 *not a critical value,* S_{λ_0} *has* k *non-vanishing principal curvatures at every point. Assume that* $V\in l^{(k+2)/2}(\mathbb{Z}^d)$*. Then the following hold.*

- *For any sequence* $(z_n)_{n\in\mathbb{N}}\subset\sigma_{\mathrm{d}}(T(p)+V)$ *accumulating to a* λ_0 *we have* $\mathrm{dist}(z_n,\lambda_0)\in l^1(\mathbb{N})$*.*
- *If* $\|V\|_{l^{(k+2)/2}}\ll 1$*, then there are no eigenvalues (including embedded ones) near* λ_0*.*

Discrete Laplacian

$$\Delta f(x)=2df(x)-\sum_{|e|=1}f(x+e)=\sum_{j=1}^{d}(1-\cos(p_j)f(x)$$

Counterexample in the strictly convex case

Theorem

Let $T : \mathbb{T}^d \to \mathbb{R}$ be a smooth function. Then the following hold:

1. *For any λ in the interior of $\mathrm{Ran}\,(T)$, there exists a potential $V : \mathbb{Z}^d \to \mathbb{C}$, decaying as*

$$|V(x)| \leq C_\epsilon (1 + |x_1| + x_2^2 + \ldots + x_d^2)^{-1+\epsilon}$$

   for arbitrary $\epsilon > 0$ and such that λ is an eigenvalue of $T(p) + V$.

2. *For any λ in the interior of $\mathrm{Ran}\,(T)$, there exists a sequence of potentials $V_n : \mathbb{Z}^d \to \mathbb{C}$ such that λ is an eigenvalue of $T(p) + V_n$ and for any $q > (d+1)/2$, we have that $\|V_n\|_{l^q} \to 0$ as $n \to \infty$.*

Outline

Schrödinger operators with unbounded background fields

Assume A_0, V_0 smooth, real-valued, $V_0 \geq 0$,

$$|\partial^\alpha A_0(x)|\langle x\rangle^{|\alpha|-1} + |\partial^\alpha V_0(x)|\langle x\rangle^{|\alpha|-2} + |\partial^\alpha B_0(x)|\langle x\rangle^{\epsilon_\alpha - 1} \leq C_\alpha$$
$$V_1 \in L^r(\mathbb{R}^d;\mathbb{C}) \quad r \in [d/2,\infty] \text{ for } d \geq 3, \quad r \in (1,\infty] \text{ for } d = 2,$$
$$W \in L^\infty(\mathbb{R}^d;\mathbb{R}),$$
$$A_1 : \mathbb{R}^d \to \mathbb{C}^n, \quad |A_1(x)| + |\nabla A_1(x)| \leq C_{A_1}\langle x\rangle^{-1-\mu}.$$

Theorem (Cuenin, Kenig 2017)

Let $d \geq 2$. There exists $\epsilon_0 > 0$ s.t. whenever $C_{A_1} < \epsilon_0$, then

$$\sigma(H) \subset \left\{ |\operatorname{Im} z|^{1-\frac{d}{2r}} \lesssim 1 + \frac{1 + \|W\|_\infty}{1 - C_{A_1}/\epsilon_0} \|V_1\|_{L^r(\mathbb{R}^d)} \right\}$$

Landau Hamiltonian

Consider the Landau Hamiltonian in $d = 2$,

$$H = \left(-\mathrm{i}\frac{\partial}{\partial x} + \frac{y}{2}\right)^2 + \left(-\mathrm{i}\frac{\partial}{\partial y} - \frac{x}{2}\right)^2 + V(x,y), \quad (x,y) \in \mathbb{R}^2.$$

Its spectrum consists of eigenvalues $\lambda_k = 2k+1$, $k = 0, 1, \ldots$, of infinite multiplicity. Then

$$\sigma(H) \cap \{z : |z - \lambda_k| \leq 1/2\} \subset \left\{z : |z - \lambda_k| \lesssim \|V\|_{L^r(\mathbb{R}^d)} \lambda_k^{\nu(r)}\right\},$$

where

$$\nu(r) := \begin{cases} \frac{1}{r} - 1, & 1 \leq r \leq \frac{3}{2}, \\ -\frac{1}{2r} & \frac{3}{2} \leq r \leq \infty. \end{cases}$$

- This uses sharp results of Koch and Ricci (2007) on L^p bounds of eigenfunctions.