On the Dirac equation with potential and the Lochak–Majorana condition

Piero D'Ancona

Dipartimento di Matematica
Sapienza Università di Roma

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References

Partly based on joint works with Mamoru Okamoto (Shinshu University, Nagano) and Federico Cacciafesta (Milano Bicocca)

- D.: Kato smoothing and Strichartz estimates for wave equations with magnetic potentials, *CMP* 2015
- D.-Cacciafesta: Endpoint estimates and global existence for the nonlinear Dirac equation with potential, JDE 2013
- D.-Okamoto: Blowup and ill-posedness results for the Dirac equation without gauge invariance, *EECT* 2016
- D.-Okamoto: On the cubic Dirac equation with potential and the Lochak–Majorana condition, in progress

Introduction

Notations: the Dirac operator is the operator on $L^2(\mathbb{R}^3;\mathbb{C}^4)$

$$\mathcal{D} = i^{-1}\left(\alpha_1\frac{\partial}{\partial x_1} + \alpha_2\frac{\partial}{\partial x_2} + \alpha_2\frac{\partial}{\partial x_3}\right) = i^{-1}\alpha\cdot\partial$$

where α_j are the Dirac matrices

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$\mathcal{D}$ is selfadjoint, non positive, with $\sigma(\mathcal{D}) = \mathbb{R}$

Mass represented by a term $m\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$, $m \geq 0$
We have then $\sigma(\mathcal{D} + m\beta) = (-\infty, -m] \cup [m, +\infty)$

We denote the corresponding unitary group with

$$u(t,x) = e^{it\mathcal{D}} f \in C(\mathbb{R}, L^2) \qquad (f \in L^2)$$

u solves the constant coefficient, 4×4 hyperbolic system with multiplicity 2

$$iu_t + \mathcal{D}u = 0, \qquad u(0,x) = f(x) \in L^2(\mathbb{R}^3; \mathbb{C}^4)$$

The commutation identities

$$\alpha_j^* = \alpha_j, \quad \beta^* = \beta, \quad \beta^2 = I_4, \quad \beta\alpha_j + \alpha_j\beta = 0 \quad \text{for } j = 1, 2, 3,$$
$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk} I_4 \quad \text{for } j, k = 1, 2, 3,$$

imply

$$\mathcal{D}^2 = -\Delta I_4$$

Thus massless Dirac can be reduced to wave equations

$$(\mathcal{D} + i\partial_t)(\mathcal{D} - i\partial_t) = (-\Delta + \partial_{tt}^2) I_4$$

and massive Dirac to Klein-Gordon equations

$$(m\beta + \mathcal{D} + i\partial_t)(m\beta + \mathcal{D} - i\partial_t) = (m^2 - \Delta + \partial_{tt}^2) I_4$$

External potentials

QED prescriptions:

- describe the field $(\mathbf{E}, \mathbf{B})$ via the vector potential

$$A = (A_0, A_1, A_2, A_3)$$

$$\mathbf{B} = \nabla \times (A_1, A_2, A_3), \qquad \mathbf{E} = \nabla A_0 - \partial_t (A_1, A_2, A_3)$$

- introduce the field into the equations via the rule

$$\partial_\mu \to \partial_\mu^A = \partial_\mu - iA_\mu, \qquad \mu = 0, \dots, 4$$

$$\partial_t^A = \partial_t - iA_0, \qquad \mathcal{D}^A = i^{-1} \sum_{j=1}^{3} \alpha_j \partial_j^A$$

This leads naturally to the Maxwell–Dirac system, with a potential evolving in time, or the simpler Dirac-Klein-Gordon system:

$$(-\Box + M)\phi = g\,\langle \beta\psi, \psi \rangle_{\mathbb{C}^4}\,, \qquad M \geq 0,\; g > 0$$
$$(i\partial_t + \mathcal{D} + \beta m)\psi = g\phi\beta\psi, \qquad m \geq 0$$

(meson + Dirac field interacting via Yukawa coupling)

The scaling critical space is

$$(\psi_0, \phi_0, \phi_1) \in L^2 \times \dot{H}^{1/2} \times \dot{H}^{-1/2}$$

$$\psi(t,x) \longrightarrow L^{-3/2}\psi(t/L, x/L), \qquad \phi(t,x) \longrightarrow L^{-1}\phi(t/L, x/L)$$

Results available without using the algebraic structure:

- LWP in $H^{1+\epsilon} \times H^{3/2+\epsilon} \times H^{1/2+\epsilon}$: classical, energy estimates and Sobolev embeddings; GWP for small, smooth data. Bachelot 88
- LWP in $H^{1/2+\epsilon} \times H^{1+\epsilon} \times H^{\epsilon}$: Strichartz estimates. Ponce-Sideris 93

Using the null structure:

- Higher regularity $\phi \in H^2$: null structure in KG. Klainerman-Machedon 92, Beals-Bezard 96
- LWP in $H^{1/2} \times H^1 \times L^2$: null structure in Dirac squared. Bournaveas 99
- LWP in $H^{1/4+\epsilon} \times H^1 \times L^2$: same idea. Fang-Grillakis 05

Using the full algebraic structure it is possible to prove an almost optimal result:

Theorem (D.-Foschi-Selberg 07)

DKG in $1+3D$ *is LWP for* $(\psi_0, \phi_0, \phi_1) \in H^{\varepsilon} \times H^{1/2+\varepsilon} \times H^{-1/2+\varepsilon}$

Tesfahun 07: LWP in $H^s \times H^r \times H^{r-1}$ for (s, r) in an open convex region emanating from $(0, 1/2)$

Many results available in 1+1D, a few in 1+2D and for the more general Maxwell–Dirac system

The main problem

Simplified model for DKG/MD: cubic Dirac equation with potential, for a field $u(t,x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$

$$\boxed{iu_t + \mathcal{D}u + V(x)u = \langle \beta u, u \rangle \beta u, \qquad u(0,x) = f(x)}$$

where $V(x) = V(x)^*$ a Hermitian 4×4 matrix

Most of the following results hold for more general cubic (gauge invariant) nonlinearities $P_3(u)$

The ‘true’ model from QED corresponds to

$$V(x) = A_0 I_4 + A_1(x)\alpha_1 + A_2(x)\alpha_2 + A_3(x)\alpha_3$$

$A_0 \equiv 0$ is not restrictive, via the choice of gauge

$$\widetilde{u} = e^{i\chi} u, \qquad \widetilde{A}_\mu = A_\mu + \partial_\mu \chi$$

In the following I shall use A_0 with a different meaning!

The unperturbed equation

Classical unperturbed NLD (with $V(x) = 0$)

$$\boxed{iu_t + \mathcal{D}u = F(u), \qquad u(0,x) = f(x)}$$

with a homogeneous nonlinearity

$$F(u) \sim |u|^\gamma, \qquad \gamma \geq 3$$

Global existence of small H^s solutions:

- early results: Reed, Najman, Moreau, Bachelot, Dias-Figueira and others
- Escobedo-Vega 97: $\gamma > 3$ and $s > \frac{3}{2} - \frac{1}{\gamma-1}$
- Machihara-Nakamura-Ozawa 04: $\gamma = 3$ and $s > 1$

GWP for small H^1 data a long standing open problem, solved:

- Bejenaru–Herr 15: massive case $m > 0$
- Bournaveas–Candy 15: massless case $m = 0$

This kind of result seems beyond reach for $V(x) \neq 0$

However:

- Machihara-Nakamura-Nakanishi-Ozawa 05: global existence of small H^1 solutions provided the data are radial, or have some additional angular regularity

Tool: endpoint Strichartz estimate with angular regularity

Strichartz estimates

Using $(\mathcal{D} + i\partial_t)(\mathcal{D} - i\partial_t) = (-\Delta + \partial_{tt}^2)I_4$ we can represent the Dirac flow in terms of the wave flow

$$e^{it\mathcal{D}} f = \cos(t|D|)f + i\frac{\sin(t|D|)}{|D|}\mathcal{D}f, \qquad |D| = (-\Delta)^{1/2}$$

Strichartz estimates for the WE apply to Dirac

Strichartz+Sobolev gives

$$\boxed{\||D|^{\frac{n}{r}+\frac{1}{p}-\frac{n}{2}}e^{it\mathcal{D}}f\|_{L^pL^r} \lesssim \|f\|_{L^2}}$$

for all p, r such that

$$p \in [2, \infty] \qquad 0 < \frac{1}{r} \leq \frac{1}{2} - \frac{2}{(n-1)p}.$$

The endpoint $(p, r) = (2, \infty)$ is false for general data f

(Actual counterexamples are known for $n = 3$ but it should not be difficult to extend to $n \geq 4$)

The endpoint case

Klainerman-Machedon 93: endpoint estimate false for Dirac/wave

$$\|e^{it\mathcal{D}} f\|_{L^2 L^\infty} \not\lesssim \||D|f\|_{L^2} \qquad (n=3)$$

If it were true, it would give a one–line proof of GWP in H^1

- Replacing L^∞ with BMO does not help
- Restricting frequencies does not help
- Similar situation for the 2D Schrödinger equation (in the radial case, estimate false but a BMO estimate or an estimate with loss of angular regularity can be proved)
- For Schrödinger, the nonhomogeneous endpoint-endpoint estimates are false but the endpoint-non endpoint combinations are true

Klainerman-Machedon 93: for the radial 3D WE the estimate holds

$$f = f(|x|) \quad \Longrightarrow \quad \|e^{it\mathcal{D}} f\|_{L^2 L^\infty} \lesssim \||D| f\|_{L^2} \qquad (n = 3)$$

Elementary proof:

$$\frac{\sin(t|D|)}{|D|} f = \frac{c}{|x|} \int_{||x|-t|}^{|x|+t} s f(s) ds \lesssim M(g)(t)$$

$M(g)$ is the maximal function of $g(s) = sf(s)$

$$\Longrightarrow \quad \left\| \frac{\sin(t|D|)}{|D|} f \right\|_{L^\infty_x} \lesssim M(g)(t)$$

and by a standard maximal estimate

$$\left\| \frac{\sin(t|D|)}{|D|} f \right\|_{L^2_t L^\infty_x} \lesssim \|g\|_{L^2(\mathbb{R})} = \|sf(s)\|_{L^2_s(\mathbb{R})} \simeq \|f\|_{L^2(\mathbb{R}^3)}$$

Fang-Wang 06: similar results in dimension $n \geq 3$

Estimates with angular regularity

The tempting argument

radial symmetry $\Longrightarrow$ *endpoint estimate* $\Longrightarrow$ *GWP*

does not work for Dirac since radial data $\not\Longrightarrow$ radial solution

The Dirac operator $\mathcal{D}$ flips couples of harmonics with different eigenvalues. The natural decomposition is

$$L^2(\mathbb{R}^3)^4 \simeq \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} L^2(0,+\infty; r^2dr) \otimes H_{m_j k_j}$$

where the $H_{m_j k_j}$ are 2-dimensional

Machihara et al. 05:

$$\|e^{it|D|}f\|_{L^2_t L^\infty_{|x|} L^p_\omega} \lesssim \sqrt{p} \cdot \||D|f\|_{L^2} \qquad \forall p < \infty$$

for the norm (obvious modification for $a = \infty$)

$$\|f\|_{L^a_{|x|} L^b_\omega} = \left(\int_0^\infty \|f(r\,\cdot\,)\|^a_{L^b(\mathbb{S}^{n-1})} r^{n-1} dr \right)^{\frac{1}{a}}$$

Combined with Sobolev embedding on $\mathbb{S}^2$ this gives the estimate with angular (loss of) regularity

$$\|e^{it|D|}f\|_{L^2_t L^\infty_x} \lesssim C_\epsilon \||D|\Lambda^\epsilon_\omega f\|_{L^2}, \qquad \Lambda_\omega = (1 - \Delta_{\mathbb{S}^2})^{\frac{1}{2}}$$

This gives GWP for cubic NDirac for data with small $\||D|\Lambda^\epsilon_\omega f\|$ norm, which includes radial H^1 data

Estimates with angular regularity for dispersive equations:

- Hoshiro 97 (to my knowledge, first who noticed)
- Machihara-Nakamura-Nakanishi-Ozawa 05
- Sterbenz-Rodnianski 05
- Fang-Wang 06, 08
- Sogge 08
- Jiang-Wang-Yu 10

The Dirac equation with a small potential

Consider the equation

$$\boxed{iu_t + \mathcal{D}u + V(x)u = P_3(u, \overline{u}), \qquad u(0,x) = f(x)} \tag{1}$$

with $V(x) = V(x)^*$ and for some $s > 1$, $C, \delta > 0$

$$\|\Lambda_\omega^s V(|x|\,\cdot\,)\|_{L^2(\mathbb{S}^2)} \leq \frac{\delta}{v(x)}, \qquad \|\Lambda_\omega^s \nabla V(|x|\,\cdot\,)\|_{L^2(\mathbb{S}^2)} \leq \frac{C}{v(x)},$$

where $v(x) = |x|^{\frac{1}{2}} |\log|x||^{\frac{1}{2}+} + |x|^{1+}$.

In Cacciafesta-D. JDE 13 we extended the result of Machihara et al. to the case of *small* potentials V, i.e., with $\delta \ll 1$

Theorem (Cacciafesta-D. JDE 13)

Let $P_3(u,\overline{u})$ be a $\mathbb{C}^4$-valued homogeneous cubic polynomial, V as above with $\delta \ll 1$ and $s > 1$. Then for all initial data with

$$\|\Lambda_\omega^s f\|_{H^1} \ll 1$$

the Cauchy problem (1) *admits a unique global solution $u \in CH^1$. Moreover $u \in L^2L^\infty$ and $\Lambda_\omega^s u \in L^\infty H^1$.*

Main result 1: large potentials

In D.-Okamoto 17 we extend the previous result to large potentials, with sharper decay and regularity conditions

Decompose $V(x)$ as

$$\boxed{V = A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + A_0\beta + V_0}$$

with $A_j : \mathbb{R}^3 \to \mathbb{R}$ and $V_0 = V_0^* : \mathbb{R}^3 \to M_4(\mathbb{C})$

The A_j can be large, while V_0 is a small perturbation

Define the dyadic norm on $\mathbb{R}^3$

$$\|\sigma\|_{\ell^1 L^\infty} := \sum_{j\in\mathbb{Z}} \|\sigma\|_{L^\infty(2^j \leq |x| < 2^{j+1})}, \qquad \ell^p L^q \text{ similar}$$

(e.g.: $\sigma(x) = C|1 + |\log|x||^{-\epsilon} \in \ell^1 L^\infty$ for $\epsilon > 1$)

We assume that for some $s > 1$ and $\delta > 0$

- $\mathcal{D} + V$ is selfadjoint with domain $H^1(\mathbb{R}^3; \mathbb{C}^4)$
- 0 is not an eigenvalue or resonance of $\mathcal{D} + V$
- $|x| \|\Lambda^s_\omega V(|x| \cdot)\|_{L^2(\mathbb{S}^2)} + |x|\langle x\rangle \|\Lambda^s_\omega \partial V(|x| \cdot)\|_{L^2(\mathbb{S}^2)} \in \ell^1 L^\infty$
- $|x|^{1/2}|A_0| \in \ell^1 L^\infty$
- $|x||V_0| + |x|^2|V_0| \in \ell^1 L^\infty$ with norm $\delta \ll 1$

(the actual assumptions are slightly weaker)

Theorem (D.-Okamoto 2017)

Under the previous assumptions, if $\delta \ll 1$ then for all initial data with $\|\Lambda_\omega^s f\|_{H^1} \ll 1$, Problem (1) *has a unique global solution $u \in CH^1 \cap L^2 L^\infty$ with $\Lambda_\omega^s u \in L^\infty H^1$.*
Moreover u scatters to a free solution, i.e., there exists $u_+ \in \Lambda_\omega^{-s} H^1$ such that

$$\lim_{t\to\infty} \|\Lambda_\omega^s u(t) - \Lambda_\omega^s e^{it(\mathcal{D}+V)} u_+\|_{H^1} = 0,$$

and similarly for $t \to -\infty$.

Main result 2: large data

Consider the subspace of $\mathbb{C}^4$

$$E := \{z \in \mathbb{C}^4 \colon\, z_1 = \overline{z}_4, z_2 = -\overline{z}_3\} \tag{2}$$

Equivalent definition ($\gamma = i\gamma_2$):

$$z \in E \quad \Longleftrightarrow \quad \gamma z = \overline{z}, \qquad \gamma := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Lochak–Majorana condition

Let $f(x) \in L^2(\mathbb{R}^3; \mathbb{C}^4)$. For a.e. $x \in \mathbb{R}^3$,

$$f(x) \in E.$$

(More generally, $\exists \theta \in \mathbb{R}$ s.t. $e^{i\theta} f \in E$)

Two facts:

- The LM condition is preserved by the free Dirac flow: if f satisfies LM then $e^{it\mathcal{D}}f$ satisfies LM for all t
- f satisfies LM iff its chiral invariant $\rho(f)$ vanishes. Here

$$\rho(f) := |\langle \beta f, f\rangle|^2 + |\langle \alpha_5 f, f\rangle|^2, \qquad \alpha_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

As a consequence,

- If the initial data f satisfy LM then the solution $u = e^{it\mathcal{D}}f$ of the linear equation solves also the NLD: $iu_t + \mathcal{D}u = \langle \beta u, u\rangle \beta u$ ($\equiv 0$)

Bachelot 89: GWP for the cubic NLD for small H^6 perturbations of data satisfying LM

Introduce the projection $P : \mathbb{C}^4 \to E$

$$P \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} z_1 + z_4 \\ z_2 - z_3 \\ \overline{z}_3 - \overline{z}_2 \\ \overline{z}_1 + \overline{z}_4 \end{pmatrix}. \tag{3}$$

Bachelot's condition on the data can be written

$$\|(I - P)f\|_{H^6} \ll 1$$

In particular f can be large

We extend the result to a nonzero potential V such that the perturbed flow $e^{it(\mathcal{D}+V)}$ preserves the LM condition

Let $\mathcal{V}$ be the space of 4×4 matrices of the form

$$\mathcal{V} := \left\{ \begin{pmatrix} a & z & w & 0 \\ \overline{z} & b & 0 & w \\ \overline{w} & 0 & -b & z \\ 0 & \overline{w} & \overline{z} & -a \end{pmatrix} : \ a, b \in \mathbb{R}, \ z, w \in \mathbb{C} \right\}$$

We shall assume

$$V(x) \in \mathcal{V} \quad \text{for all} \quad x \in \mathbb{R}^3.$$

If $V = A \cdot \alpha + A_0\beta + V_0$ then $V \in \mathcal{V}$ implies $A_1 = A_2 = A_3 = 0$ i.e.

$$V \in \mathcal{V} \quad \iff \quad V = A_0\beta + V_0, \qquad V_0 \in \mathcal{V}$$

Theorem (D.-Okamoto 2017)

Assume V as in the previous Theorem with $\delta \ll 1$, and in addition assume $V(x) \in \mathcal{V}$ for all x. Then the conclusions of the previous Theorem are valid for all data f such that

$$\|\Lambda_\omega^s (I - P) f\|_{H^1} \ll 1.$$

Note that in

$$V = A_0 \beta + V_0$$

the component A_0 is allowed to be large

Note that the existence of large solution for LM data depends heavily on the structure of the nonlinearity

D.-Okamoto 16: If we replace $\langle \beta u, u \rangle \beta u$ with $|u|^3 I_4$, it is possible to construct LM data such that the solution blows up in a finite time, even in the case $V(x) = 0$

Sketch of proof (Thm. 1, large potentials)

Squaring $\mathcal{D} + V$ gives a system of Schrödinger operators

$$L := (\mathcal{D} + V)^2 = -I_4 \Delta_A - W - Z \cdot \partial$$

where

$$W = B - I_4 A_0^2 - \mathcal{D}\beta A_0 - \mathcal{D} V_0 - V_0^2 - V_0(\alpha \cdot A + \beta A_0) - (\alpha \cdot A + \beta A_0) V_0$$

and

$$Z_j = i(V_0 \alpha_j + \alpha_j V_0)$$

Here B represents the magnetic field:

$$B = i \sum_{j<k} B_{jk} \alpha_j \alpha_k, \qquad B_{jk} = \partial_j A_k - \partial_k B_j$$

First step: resolvent estimate

We prove a resolvent estimate for

$$R(z) = (-L - z)^{-1}$$

of the form: for all $z \in \mathbb{C}$ with $|\Im z| < 1$,

$$\|R(z)f\|_{\dot{X}} + |z|^{\frac{1}{2}}\|R(z)f\|_{\dot{Y}} + \|\partial R(z)f\|_{\dot{Y}} \leq C\|f\|_{L^2_\rho}$$

Spaces:

$$\|v\|_{\dot{Y}}^2 = \sup_{R>0} \tfrac{1}{R} \int_{|x|<R} |v|^2 dx$$

$$\|v\|_{\dot{X}}^2 = \sup_{R>0} \tfrac{1}{R^2} \int_{|x|=R} |v|^2 dS$$

$$\|v\|_{L^2_\rho}^2 = \||x|^{1/2}\rho^{-1}v\|_{L^2}$$

$\rho > 0$ is a weight in $\ell^2 L^\infty$ (e.g. $(1 + |\log|x||)^{-\epsilon}$, $\epsilon > 1/2$)

Large frequency regime $\Re z \gg 1$

We use a multiplier method which gives a sharp estimate, with explicit constants

Note that we are dealing with a system of Schrödinger equations, but with diagonal principal part

Small frequency regime $|\Re z| \leq C$

For $\Re z$ in any bounded region we use the Lippmann–Schwinger equation

$$R(z) = R_0(z)(I_4 - (W + Z \cdot \partial)R_0(z))^{-1}$$

where R_0 is the free resolvent $R_0(z) = I_4(-\Delta - z)^{-1}$.

The operator

$$(W + Z \cdot \partial)R_0(z) : L^2_\rho \to L^2_\rho$$

is compact and $I_4 - (W + Z \cdot \partial)R_0(z)$ can be inverted with a locally uniform bound on the inverse, via analytic Fredholm theory

A crucial step is proving that $I_4 - (W + Z \cdot \partial)R_0(z)$ is injective, i.e., L has no embedded eigenvalues or resonances

For $\Re z > 0$ this follows by an application of Koch–Tataru 06 (Carleman estimates and absence of embedded eigenvalues)

For $z = 0$ this is an explicit assumption

Second step: Kato theory for WE

Kato 65: a uniform resolvent estimate

$$\|AR(z)A^* f\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}} \tag{4}$$

is equivalent to a smoothing estimate

$$\|Ae^{itL} f\|_{L^2_t \mathcal{H}} \lesssim \|f\|_{\mathcal{H}}$$

D. 2015: estimate (4) implies also the smoothing estimate for the wave flow

$$\|Ae^{it\sqrt{L}} f\|_{L^2_t \mathcal{H}} \lesssim \|L^{1/4} f\|_{\mathcal{H}}$$

This is almost an estimate for $e^{it(\mathcal{D}+V)}$ since $\sqrt{L} \neq (\mathcal{D}+V)$. Some more spectral theory gives the smoothing estimate

$$\||x|^{-1/2} \rho e^{it(\mathcal{D}+V)} f\|_{L^2_t L^2_x} \lesssim \|f\|_{L^2}, \qquad \rho \in \ell^2 L^\infty$$

Third step: Strichartz/smooothing estimate

By a minor modification of a result of Cacciafesta–D. 13 we obtain the mixed endpoint Strichartz–smoothing estimate

$$\|\Lambda_\omega^s \int_0^t e^{i(t-t')\mathcal{D}} F dt'\|_{L_t^2 L_{|x|}^\infty L_\omega^2} \lesssim \|\rho^{-1}|x|^{\frac{1}{2}}|D|\Lambda_\omega^s F\|_{L_t^2 L^2}$$

Combining this with the smoothing estimate for $e^{it(\mathcal{D}+V)}$ we get an endpoint Strichartz estimate for the perturbed flow

$$\|\Lambda_\omega^s e^{it(\mathcal{D}+V)} f\|_{L_t^2 L^\infty L^2} + \|\Lambda_\omega^s e^{it(\mathcal{D}+V)} f\|_{L_t^2 H^1} \lesssim \|\Lambda_\omega^s f\|_{H^1}$$

GWP and scattering for small data follow easily

Sketch of proof (Thm. 2, large data)

Assume $V(x)$ has the special structure

$$\begin{pmatrix} a & z & w & 0 \\ \overline{z} & b & 0 & w \\ \overline{w} & 0 & -b & z \\ 0 & \overline{w} & \overline{z} & -a \end{pmatrix} \quad \text{for some } a, b \in \mathbb{R} \text{ and } z, w \in \mathbb{C}$$

(i.e., $V(x) \in \mathcal{V}$ for all x)
Then the LM condition is preserved by the perturbed flow $e^{it(\mathcal{D}+V)}$

Thus if the data $\chi_0 \in L^2$ satisfy LM, the corresponding solution of the linear equation solves also the NLD:

$$\chi = e^{it(\mathcal{D}+V)}\chi_0 \quad \Longrightarrow \quad i\chi_t + (\mathcal{D}+V)\chi = \langle \beta\chi, \chi\rangle \beta\chi \quad (\equiv 0)$$

If f are arbitrary data, the projection $\chi_0 = Pf$ on E satisfies LM and generates a global large reference solution $\chi(t,x)$ of NLD with potential

Under the assumption

$$\|\Lambda_\omega^s (I-P) f\|_{H^1} \ll 1$$

i.e. f close enough to LM data, we prove that the corresponding solution u remains close to the reference solution χ for all times

Since the reference solution χ is large, we must split $[0, +\infty)$ in a finite number of intervals

$$[0, +\infty) = [0, T_1] \cup [T_1, T_2] \cup \ldots [T_{N-1}, T_N] \cup [T_N, +\infty)$$

such that the Strichartz norm of χ is sufficiently small in each one, and then a continuation argument gives the result