On the eigenvalues of perturbed projected Coulomb–Dirac operators

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Linear and Nonlinear Dirac Equation: advances and open problems

Como

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References

This talk is based on joint results with

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Lieb-Thirring and Cwickel-Lieb-Rozenblum inequalities for perturbed graphene with a Coulomb impurity

Accepted to J. Spectr. Theory. arXiv:1603.01485,

On the virtual levels of positively projected massless Coulomb-Dirac operators

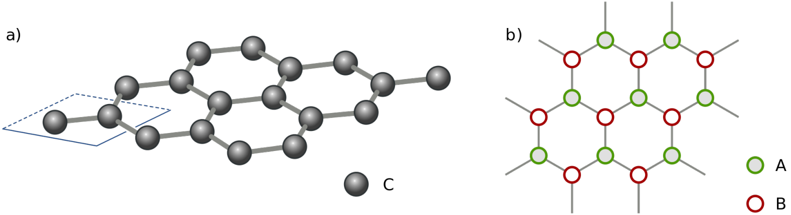
arXiv:1607.08902

and

Lower bounds on the moduli of three-dimensional Coulomb-Dirac operators via fractional Laplacians with applications

arXiv:1607.08902

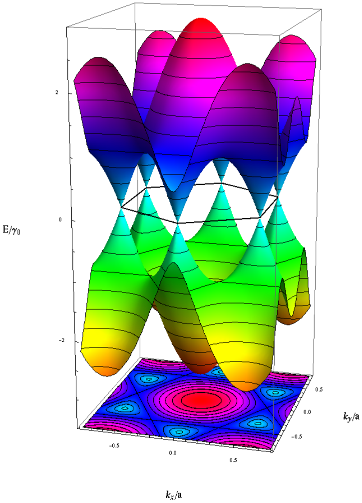
Graphene and its effective Hamiltonian



The effective Hamiltonians for graphene are given by:

1. The nearest neighbour lattice Laplacian on a two-dimensional honeycomb lattice (Wallace '47, Pereira, Nilson, Castro Neto '07, Castro Neto, Guinea, Peres, Novoselov, Geim '09);
2. Periodic Schrödinger operator $-\Delta + W$, where W is smooth and has honeycomb symmetry (Feffermann, Weinstein '12).

Graphene and its effective Hamiltonian



Energy of electrons in graphene in the tight-binding model, The Band Theory of Graphite
P. R. Wallace, Phys. Rev. 71, 622, 1 May 1947, http://dx.doi.org/10.1103/PhysRev.71.622,
Paul Wenk, Wikimedia Commons

Near the conical point the effective Hamiltonain is given in $\mathsf{L}^2(\mathbb{R}^2, \mathbb{C}^2)$ by the massless Dirac operator

$$D_0 := v_F(-\mathrm{i}\hbar\nabla \cdot \boldsymbol{\sigma}),$$

with

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \right)$$

and $v_F \approx 10^6$ m/s ($\approx 0.003c$). We choose units with $v_F\hbar = 1$.

Coulomb-Dirac operator and the Dirac sea

Suppose now that the graphene sheet contains an attractive Coulomb impurity of strength ν. The effective Hamiltonian is then formally given by

$$D_\nu := -\mathrm{i}\nabla \cdot \boldsymbol{\sigma} - \nu|\cdot|^{-1}.$$

- It turns out that there exists a "distinguished" self-adjoint realisation of D_ν for $|\nu| \leqslant 1/2$.
- D_ν is invariant with respect to rotations and scaling in the plane.
- The space of physically available states is not $\mathsf{L}^2(\mathbb{R}^2, \mathbb{C}^2)$, but $P_+^\nu \mathsf{L}^2(\mathbb{R}^2, \mathbb{C}^2)$, where $P_+^\nu := P_{[0,\infty)}(D_\nu)$ is the spectral projection of D_ν to the interval $[0, \infty)$.

Perturbed positively projected Coulomb-Dirac Operator

- We now want to apply a further Hermitian matrix-valued potential V. If V is not strong enough to substantially modify the Dirac sea, the effective Hamiltonian takes the form

$$D_\nu(V) := P_+^\nu (D_\nu - V) P_+^\nu.$$

- We assume that

$$\mathrm{tr}(V_+^{2+\gamma}) \in \mathsf{L}^1(\mathbb{R}^2) \text{ with } (\nu, \gamma) \in \big([0, 1/2] \times [0, \infty)\big) \setminus \big\{(1/2, 0)\big\},$$

where

$$x_\pm := \max\{\pm x, 0\}.$$

- Under this assumption (and form-boundedness of V_- with respect to D_ν) $D_\nu(V)$ is self-adjoint, with negative spectrum consisting of eigenvalues possibly accumulating at zero.
- Our main results provide estimates of these eigenvalues.

The Cwikel-Lieb-Rosenblum bound on the number of negative eigenvalues

Theorem 1

Let $\nu \in [0, 1/2)$*. There exists* $C_\nu^{\mathrm{CLR}} > 0$ *such that*

$$\operatorname{rank}\big(D_\nu(V)\big)_- \leqslant C_\nu^{\mathrm{CLR}} \int_{\mathbb{R}^2} \operatorname{tr}\big(V_+(\mathbf{x})\big)^2 \mathrm{d}\mathbf{x}. \tag{1}$$

Analogues of Theorem 1 are known for many bounded from below self-adjoint operators as Cwickel-Lieb-Rozenblum inequalities. In particular, in Frank '14 it is proved that the estimate

$$\operatorname{rank}\big((-\Delta)^s - V\big)_- \leqslant (4\pi s)^{-1}(1-s)^{(s-2)/s} \int_{\mathbb{R}^2} \operatorname{tr}\big(V_+(\mathbf{x})\big)^{1/s} \mathrm{d}\mathbf{x}$$

holds for all $0 < s < 1$.

Virtual level at zero

Theorem 2

Let

$$\widetilde{V}(r) := \frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix} V_{11}(r,\varphi) & -\mathrm{i}V_{12}(r,\varphi)\mathrm{e}^{\mathrm{i}\varphi} \\ \mathrm{i}V_{21}(r,\varphi)\mathrm{e}^{-\mathrm{i}\varphi} & V_{11}(r,\varphi)\end{pmatrix}\,\mathrm{d}\varphi.$$

Suppose that

$$\|\widetilde{V}\|_{\mathbb{C}^{2\times 2}} \in \mathsf{L}^1\big(\mathbb{R}_+,(1+r^2)\mathrm{d}r\big)$$

and

$$\int_0^\infty \left\langle \begin{pmatrix}-1\\1\end{pmatrix}, \widetilde{V}(r)\begin{pmatrix}-1\\1\end{pmatrix}\right\rangle_{\mathbb{C}^2}\mathrm{d}r > 0.$$

Then the negative spectrum of $D_{1/2}(V)$ is non-empty.

(Hardy-)Lieb-Thirring inequalities

Theorem 3

Let $\nu \in [0, 1/2]$ *and* $\gamma > 0$. *There exists* $C^{\mathrm{LT}}_{\nu,\gamma} > 0$ *such that*

$$\mathrm{tr}\left(D_\nu(V)\right)_-^\gamma \leqslant C^{\mathrm{LT}}_{\nu,\gamma} \int_{\mathbb{R}^2} \mathrm{tr}\left(V_+(\mathbf{x})\right)^{2+\gamma} \mathrm{d}\mathbf{x}. \tag{2}$$

Theorem 3 is a form of Lieb-Thirring inequality.

Hardy-Lieb-Thirring inequalities

For $\nu = 1/2$ Theorem 3 is an equivalent of Hardy-Lieb-Thirring inequality by Ekholm, Frank, Lieb, Seiringer:
For $d \in \mathbb{N}$ and $0 < s < d/2$ the operator

$$(-\Delta)^s - \alpha|\cdot|^{-2s}$$

is bounded below if and only if

$$\alpha \leqslant C_{s,d} := 2^{2s} \frac{\Gamma^2\big((d+2s)/4\big)}{\Gamma^2\big((d-2s)/4\big)}$$

holds. For $\gamma > 0$ there exists $L_{\gamma,d,s} > 0$ such that

$$\operatorname{tr}\big((-\Delta)^s - C_{s,d}|\cdot|^{-2s} - V\big)_-^\gamma \leqslant L_{\gamma,d,s} \int_{\mathbb{R}^d} V(\mathbf{x})_+^{\gamma+d/2s} \mathrm{d}\mathbf{x}.$$

Lower bounds via fractional Laplacian

The proofs of Theorems 1 and 3 are based upon

Theorem 4

1. *For every* $\nu \in [0, 1/2)$ *there exists* $C_\nu > 0$ *such that*

$$|D_\nu| \geqslant C_\nu \sqrt{-\Delta} \otimes \mathbb{1}_2 \tag{3}$$

   holds.

2. *For any* $\lambda \in [0, 1)$ *there exists* $K_\lambda > 0$ *such that*

$$|D_{1/2}| \geqslant \left(K_\lambda \ell^{\lambda-1}(-\Delta)^{\lambda/2} - \ell^{-1}\right) \otimes \mathbb{1}_2 \tag{4}$$

   holds for any $\ell > 0$.

The operator inequality (4)

$$|D_{1/2}| \geqslant \big(K_\lambda \ell^{\lambda-1}(-\Delta)^{\lambda/2} - \ell^{-1}\big) \otimes \mathbb{1}_2$$

is related to the estimate for the fractional Schrödinger operator with Coulomb potential in $\mathsf{L}^2(\mathbb{R}^2)$: For any $t \in (0, 1/2)$ there exists $M_t > 0$ such that

$$(-\Delta)^{1/2} - \frac{2\big(\Gamma(3/4)\big)^2}{\big(\Gamma(1/4)\big)^2 |\cdot|} \geqslant M_t \ell^{2t-1}(-\Delta)^t - \ell^{-1}$$

holds for all $\ell > 0$, see Frank '09 (and Solovej, Sørensen and Spitzer '10 for an analogous result in three dimensions).

Coulomb-Dirac operators on the half-line

For $\kappa, \nu \in \mathbb{R}$ let

$$\beta := \sqrt{\kappa^2 - \nu^2} \in \overline{\mathbb{R}_+} \cup \mathrm{i}\mathbb{R}_+.$$

and consider the differential expression

$$d^{\nu,\kappa} := \begin{pmatrix} -\nu/r & -\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} \\ \frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} & -\nu/r \end{pmatrix}. \tag{5}$$

It turns out that for $\beta \geqslant 1/2$ the corresponding symmetric operator defined on $C_0^\infty(\mathbb{R}_+, \mathbb{C}^2)$ is essentially self-adjoint in $L^2(\mathbb{R}_+, \mathbb{C}^2)$. For all other values of β there exists a one-parametric family of self-adjoint extensions $\{D^\theta_{\nu,\kappa}\}_{\theta\in[0,\pi)}$.
There exists a unitary $\mathcal{A} : L^2(\mathbb{R}^2, \mathbb{C}^2) \to \bigoplus_{\varkappa\in\mathbb{Z}+1/2} L^2(\mathbb{R}_+, \mathbb{C}^2)$ such that

$$D_\nu = \mathcal{A}^* \Bigg(\bigoplus_{\kappa\in\mathbb{Z}+1/2} D^{\nu,\kappa}_{\boldsymbol{\theta}(\kappa)} \Bigg) \mathcal{A}. \tag{6}$$

Results for $D^{\theta}_{\nu,\kappa}$

With $P_+^{\nu,\kappa,\theta} := P_{[0,\infty)}(D^{\theta}_{\nu,\kappa})$ consider the negative spectrum of

$$D^{\theta}_{\nu,\kappa}(V) := P_+^{\nu,\kappa,\theta}(D^{\theta}_{\nu,\kappa} - V)P_+^{\nu,\kappa,\theta}$$

on $P_+^{\nu,\kappa,\theta}\mathsf{L}^2(\mathbb{R}_+, \mathbb{C}^2)$. We observe the following situations:

Cases for $D^{\theta}_{\nu,\kappa}$

VL: There exists a measurable function $A^{\theta}_{\nu,\kappa}:\mathbb{R}_+\to\mathbb{C}^2$ vanishing almost nowhere such that for any V satisfying

$$\int_0^\infty \left\langle A^{\theta}_{\nu,\kappa}(r), V(r)A^{\theta}_{\nu,\kappa}(r)\right\rangle_{\mathbb{C}^2}\mathrm{d}r > 0$$

the operator $D^{\theta}_{\nu,\kappa}(V)$ has non-empty negative spectrum.

E1: For $q>1$ there exist weight functions $W^{\theta,q}_{\nu,\kappa}:\mathbb{R}_+\to\overline{\mathbb{R}_+}$ such that

$$\operatorname{rank} P_{(-\infty,0)}\big(D^{\theta}_{\nu,\kappa}(V)\big) \leqslant \int_0^\infty \big\|V_+(r)\big\|^q_{\mathbb{C}^{2\times 2}} W^{\theta,q}_{\nu,\kappa}(r)\,\mathrm{d}r.$$

E2: For $V_+\in \mathsf{L}^\infty(\mathbb{R}_+,\mathbb{C}^{2\times 2})$ there exists $K_{\nu,\kappa}\in\mathbb{R}_+$ such that

$$\operatorname{rank} P_{(-\infty,0)}\big(D^{\theta}_{\nu,\kappa}(V)\big) \leqslant K_{\nu,\kappa}\int_0^\infty \big\|V_+(r)\big\|_{\mathbb{C}^{2\times 2}} \times \Big(\ln^2(\mathrm{e}^{\tan\theta} r) + \ln^2\left(\mathrm{e} + 2r\|V_+\|_{\mathsf{L}^\infty(\mathbb{R}_+,\mathbb{C}^{2\times 2})}\right)\Big)\,\mathrm{d}r.$$

Results for $D^{\theta}_{\nu,\kappa}$

With

$$\beta := \sqrt{\kappa^2 - \nu^2} \in \overline{\mathbb{R}_+} \cup \mathrm{i}\mathbb{R}_+.$$

we get

	$\theta = 0$	$\theta = \pi/2$	$\theta \in (0,\pi) \setminus \{\pi/2\}$
$\beta \geqslant 1/2$	—	E1	—
$\beta \in (0, 1/2)$	VL	E1	E1
$\beta = 0 \neq \kappa$	E2	VL	E2
$\beta = 0 = \kappa$	VL	VL	VL
$\beta \in \mathrm{i}\mathbb{R}_+$	VL	VL	VL

Spectral representation

Theorem 5

Let Λ *be the operator of multiplication by the independent variable in* $\mathsf{L}^2(\mathbb{R},\mathbb{C},\mathrm{d}x)$. *Let* $\Phi_{0,\theta}^{\nu,\kappa}(\lambda;\cdot)$ *be the solution of* $d^{\nu,\kappa}\Phi_{0,\theta}^{\nu,\kappa} = \lambda\Phi_{0,\theta}^{\nu,\kappa}$ *satisfying the boundary condition at zero. We find an explicit* $m_\theta^{\nu,\kappa}(\lambda)$ *such that*

$$\mathcal{U}_\theta^{\nu,\kappa} : \mathsf{L}^2(\mathbb{R}_+,\mathbb{C}^2,\mathrm{d}r) \to \mathsf{L}^2(\mathbb{R},\mathbb{C},\mathrm{d}x),$$

$$(\mathcal{U}_\theta^{\nu,\kappa} f)(\lambda) := \mathsf{L}^2\text{-}\lim_{R\to\infty}\sqrt{m_\theta^{\nu,\kappa}(\lambda)}\int_{1/R}^{R}\left(\Phi_{0,\theta}^{\nu,\kappa}(\lambda;y)\right)^{\mathsf{T}} f(y)\,\mathrm{d}y$$

is well-defined and unitary. It delivers the spectral representation of $D_\theta^{\nu,\kappa}$, *i.e.*

$$D_\theta^{\nu,\kappa} = (\mathcal{U}_\theta^{\nu,\kappa})^*\Lambda\,\mathcal{U}_\theta^{\nu,\kappa}$$

holds.

Mellin transform

Let $\mathcal{M}$ be the unitary Mellin transform, first defined on $\mathsf{C}_0^\infty(\mathbb{R}_+)$ by

$$(\mathcal{M}\psi)(s) := \frac{1}{\sqrt{2\pi}} \int_0^\infty r^{-1/2-\mathrm{i}s} \psi(r) \mathrm{d}r, \tag{7}$$

and then extended to a unitary operator $\mathcal{M} : \mathsf{L}^2(\mathbb{R}_+) \to \mathsf{L}^2(\mathbb{R})$.

Definition 6

For $\lambda \in \mathbb{R} \setminus \{0\}$ let $\mathfrak{D}_\lambda$ be the set of functions $\psi \in \mathsf{L}^2(\mathbb{R})$ such that there exists Ψ analytic in the strip $\mathfrak{S}^\lambda := \big\{ z \in \mathbb{C} : \operatorname{Im} z/\lambda \in (0,1) \big\}$ with the properties

1. $\mathsf{L}^2\text{-}\lim\limits_{t \to +0} \Psi(\cdot + \mathrm{i}t\lambda) = \psi(\cdot)$;
2. there exists $\mathsf{L}^2\text{-}\lim\limits_{t \to 1-0} \Psi(\cdot + \mathrm{i}t\lambda)$;
3. $\sup\limits_{t \in (0,1)} \int_{\mathbb{R}} \big|\Psi(s + \mathrm{i}t\lambda)\big|^2 \mathrm{d}s < \infty$.

For $\lambda \in \mathbb{R}$ let the operator of multiplication by r^λ in $\mathsf{L}^2(\mathbb{R}_+, \mathrm{d}r)$ be defined on its maximal domain $\mathsf{L}^2\big(\mathbb{R}_+, (1 + r^{2\lambda})\mathrm{d}r\big)$. Applying a lemma of Titchmarsh to justify the translations of the integration contour between different values of t under Assumption 3 of Definition 6 we obtain

Theorem 7

Let $\lambda \in \mathbb{R} \setminus \{0\}$. Then the identity

$$\mathfrak{D}_\lambda = \mathcal{M}\mathsf{L}^2\big(\mathbb{R}_+, (1 + r^{2\lambda})\mathrm{d}r\big)$$

holds, and for any $\psi \in \mathfrak{D}_\lambda$ the function Ψ from Definition 6 satisfies

$$\Psi(z) = (\mathcal{M} r^{\operatorname{Im} z} \mathcal{M}^* \psi)(\operatorname{Re} z), \qquad \textit{for all } z \in \mathfrak{S}^\lambda.$$

We conclude that r^λ acts as a complex shift in the Mellin space. Indeed, for $\lambda \in \mathbb{R}$ let $R^\lambda : \mathfrak{D}_\lambda \to \mathsf{L}^2(\mathbb{R})$ be the linear operator defined by

$$R^\lambda \psi := \begin{cases} \mathsf{L}^2\text{-}\lim\limits_{t \to 1-0} \Psi(\cdot + \mathrm{i} t\lambda), & \lambda \neq 0; \\ \psi, & \lambda = 0, \end{cases}$$

with Ψ as in Definition 6. It follows from Theorem 7 that R^λ is well-defined and that

$$\mathcal{M} r^\lambda \mathcal{M}^* = R^\lambda \tag{8}$$

holds.

Fourier-Mellin theory of the relativistic massless Coulomb operator in two dimensions

We find that

$$\mathcal{T}\big((-\Delta)^{1/2} - \alpha|\cdot|^{-1}\big)\mathcal{T}^* = \bigoplus_{m\in\mathbb{Z}} \big(1 - \alpha V_{|m|-1/2}(\cdot + \mathrm{i}/2)\big) R^1,$$

where $\mathcal{T} = \mathcal{M}\mathcal{W}\mathcal{F}$ is unitary and

$$V_j(z) := \frac{\Gamma\big((j+1+\mathrm{i}z)/2\big)\Gamma\big((j+1-\mathrm{i}z)/2\big)}{2\Gamma\big((j+2+\mathrm{i}z)/2\big)\Gamma\big((j+2-\mathrm{i}z)/2\big)}, \tag{9}$$

for $j \in \mathbb{N}_0 - 1/2$ and $z \in \mathbb{C} \setminus \mathrm{i}(\mathbb{Z} + 1/2)$.
An analogous representation was used in three dimensions by Yaouanc, Oliver, and Raynal '97.

Thank you for your attention!