

# Stability of nonlinear waves in integrable Hamiltonian PDEs

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# I. Integrable Hamiltonian PDEs

An abstract Hamiltonian PDE can be written in the form

$$\frac{du}{dt} = J H'(u), \quad u(t) \in X$$

where  $X \subset L^2$  is the phase space,  $J^* = -J$  represents the symplectic structure, and  $H : X \rightarrow \mathbb{R}$  is the Hamilton function.

**Example:** Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad u(t, x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

Hamiltonian system in the form

$$\frac{du}{dt} = \frac{\partial}{\partial x} \frac{\delta H}{\delta u}, \quad \text{where} \quad H(u) = \frac{1}{2} \int_{\mathbb{R}} \left[ \left( \frac{\partial u}{\partial x} \right)^2 - 2u^3 \right] dx.$$

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where  $X \subset L^2$  is the phase space,  $J^* = -J$  represents the symplectic structure, and  $H : X \rightarrow \mathbb{R}$  is the Hamilton function.

**Example:** nonlinear Schrödinger (NLS) equation

$$i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2|u|^2 u = 0, \quad u(t, x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$$

Hamiltonian system in the form

$$\frac{du}{dt} = i \frac{\delta H}{\delta \bar{u}}, \quad \text{where} \quad H(u) = \frac{1}{2} \int_{\mathbb{R}} \left[ \left| \frac{\partial u}{\partial x} \right|^2 - 2|u|^4 \right] dx.$$

# Class of integrable Hamiltonian PDEs

## Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad u(t, x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

is integrable in the sense of the inverse scattering transform method

- The (smooth) solution  $u(t, x)$  is a potential of the Lax operator pair

$$L(u)\psi = \lambda\psi, \quad \frac{\partial \psi}{\partial t} = A(u, \lambda)\psi,$$

such that  $\lambda$  is  $(t, x)$ -independent. The Cauchy problem can be solved by a sequence of direct and inverse scattering transforms.

- Infinitely many conserved quantities exist for smooth solutions.
- Bäcklund–Darboux transformation allows to construct many exact solutions (solitary waves, periodic waves, rogue waves, etc.)

Ablowitz–Kaup–Newell–Segur, Zakharov–Shabat, Fokas, +  $\infty$ .

# New developments for integrable Hamiltonian PDEs

Many classical PDE problems, which were opened in the functional-analytic framework, have been recently solved for the integrable nonlinear PDEs.

## Example 1 : Global existence for the derivative NLS equation

$$\begin{cases} iu_t + u_{xx} + i(|u|^2u)_x = 0, & t > 0, \\ u|_{t=0} = u_0 \in X, \end{cases}$$

where  $X$  is some Banach space.

### Definition

The Cauchy problem is **locally well-posed** in  $X$  if there exists a unique solution  $u(t, \cdot) \in X$  for  $t \in (-T, T)$  with finite  $T > 0$  and the solution map  $u_0 \mapsto u(t, \cdot)$  is continuous. It is **globally well-posed** if  $T$  can be arbitrarily large.

## Example 1: Global existence for the DNLS equation

- Tsutsumi & Fukuda (1980) established local well-posedness in  $H^s(\mathbb{R})$  with  $s > \frac{3}{2}$  and extended solutions globally in  $H^2(\mathbb{R})$  for small data in  $H^1(\mathbb{R})$
- Hayashi (1993) used gauge transformation of DNLS to a system of semi-linear NLS and established local and global well-posedness in  $H^1(\mathbb{R})$  under the constraint  $\|u_0\|_{L^2} < \sqrt{2\pi}$ .
- Global existence was proved in  $H^s(\mathbb{R})$  for  $s > \frac{32}{33}$  (Takaoka, 2001),  $s > \frac{1}{2}$  (Colliander et al, 2002), and  $s = \frac{1}{2}$  (Mio-Wu-Xu, 2011) under the same constraint  $\|u_0\|_{L^2} < \sqrt{2\pi}$ .

### Recent development:

global existence without restriction on the  $L^2(\mathbb{R})$  norm.

Liu–Perry–Sulem (2016); P–Shimabukuro (2017).

# New developments for integrable nonlinear PDEs

## Example 2 : Orbital stability in spaces of low regularity

$$\begin{cases} iu_t + u_{xx} + |u|^2u = 0, & t > 0, \\ u|_{t=0} = u_0 \in X. \end{cases}$$

The Cauchy problem is globally well-posed for  $X = L^2(\mathbb{R})$  (Tsutsumi, 1986).

The family of stationary solitary waves

$$u_\omega(t, x) := \sqrt{2\omega} \operatorname{sech}(\sqrt{\omega}x) e^{i\omega t},$$

where  $\omega > 0$  is arbitrary parameter.

### Definition

The solitary wave  $u_\omega$  is said to be **orbitally** stable in  $X$  if for any  $\epsilon > 0$  there is a  $\delta > 0$  such that if  $\|u(0, \cdot) - u_\omega(0, \cdot)\|_X < \delta$  then

$$\inf_{\theta \in \mathbb{R}} \|u(t, \cdot) - e^{i\theta} u_\omega(t, \cdot)\|_X < \epsilon \quad \text{for all } t > 0.$$

## Example 2 : Orbital stability in spaces of low regularity

- Orbital stability in  $H^1(\mathbb{R})$  is proved with the energy method (Lyapunov functions and constrained minimization) Weinstein (1985), Shatah–Strauss (1985), Grillakis *et al.* (1987).
- Energy methods do not work in  $L^2(\mathbb{R})$  due to lack of control.
- With the Bäcklund–Darboux transformation, orbital and asymptotic stability of solitary waves can be obtained for the NLS equation. Mizumachi–P. (2012); Cuccagna–P. (2014); Contreras–P (2014).



# New developments for integrable nonlinear PDEs

## Example 3 : stability of non-stationary solutions

- $N$ -soliton solutions are orbitally stable in  $H^N(\mathbb{R})$ 
  - ▶ KdV [Sachs - Maddocks (1993)]
  - ▶ NLS [Kapitula (2006)]
  - ▶ Derivative NLS [Le Coz–Wu (2016)]
- Breathers are orbitally stable in  $H^2(\mathbb{R})$ 
  - ▶ modified KdV [Alejo–Munoz (2013)]
  - ▶ sine-Gordon [Alejo–Munoz (2016)]

In the rest of my talk, I will restrict attention to stability of relative equilibria in Hamiltonian systems (solitary waves, periodic waves) by using energy methods.

## II. Stability of relative equilibria in Hamiltonian systems

Consider again an abstract Hamiltonian dynamical system

$$\frac{du}{dt} = J H'(u), \quad u(t) \in X$$

where  $X \subset L^2$  is the phase space,  $J$  is a skew-adjoint operator with a bounded inverse, and  $H : X \rightarrow \mathbb{R}$  is the Hamilton function.

- Assume existence of the equilibrium  $u_0 \in X$  such that  $H'(u_0) = 0$ .
- Perform linearization  $u(t) = u_0 + ve^{\lambda t}$ , where  $\lambda$  is the spectral parameter and  $v \in X$  satisfies the spectral problem

$$JH''(u_0)v = \lambda v,$$

where  $H''(u_0) : X \rightarrow L^2$  is a self-adjoint Hessian operator.

## Main Question

Consider the spectral problem:

$$JH''(u_0)v = \lambda v, \quad v \in X.$$

**Question:** Is there a relation between unstable eigenvalues of  $JH''(u_0)$  and eigenvalues of  $H''(u_0)$ ?

**Assumptions of the negative index theory:**

- The spectrum of  $H''(u_0)$  is strictly positive except for finitely many negative and zero eigenvalues of finite multiplicity.
- The spectrum of  $JH''(u_0)$  is purely imaginary except for finitely many unstable eigenvalues.
- Multiplicity of the zero eigenvalue of  $JH''(u_0)$  is given by the number of parameters in  $u_0$  (symmetries).

## Answer for gradient systems

For a gradient system:

$$\frac{du}{dt} = -F'(u) \quad \Rightarrow \quad \lambda v = -F''(u_0)v,$$

there exists the exact relation between unstable eigenvalues of  $-F''(u_0)$  and negative eigenvalues of  $F''(u_0)$ .

### Theorem

*The number of unstable eigenvalues of  $-F''(u_0)$  is equal to the number of negative eigenvalues of  $F''(u_0)$ .*

What is about Hamiltonian systems?

$$\lambda v = JH''(u_0)v, \quad v \in X.$$

**Quadruple Symmetry:** If  $\lambda$  is an eigenvalue, so is  $-\lambda$ ,  $\bar{\lambda}$ , and  $-\bar{\lambda}$ .

# Stability Theorems for Hamiltonian Systems

Consider the spectral stability problem:

$$JH''(u_0)v = \lambda v, \quad v \in X,$$

under the assumptions above on  $J$  and  $H''(u_0)$ .

## Orbital Stability Theorem [Grillakis–Shatah–Strauss (1990)]

- Assume no symmetries/zero eigenvalues of  $H''(u_0)$ . If  $H''(u_0)$  has no negative eigenvalues, then  $JH''(u_0)$  has no unstable eigenvalues and  $u_0$  is linearly and nonlinearly stable.
- Assume zero eigenvalue of  $H''(u_0)$  of multiplicity  $N$  and related  $N$  symmetries/conserved quantities. If  $H''(u_0)$  has no negative eigenvalues under  $N$  constraints, then  $JH''(u_0)$  has no unstable eigenvalues and  $u_0$  is orbitally stable.

## Negative Index Theorem [Kapitula–Kevrekidis–Sandstede (2004)]

Assume no symmetries/zero eigenvalues of  $H''(u_0)$ . Then,

$$N_{\text{re}}(JH''(u_0)) + 2N_{\text{c}}(JH''(u_0)) + 2N_{\text{im}}^-(JH''(u_0)) = N_{\text{neg}}(H''(u_0)) < \infty,$$

where

- $N_{\text{re}}$  - number of real unstable eigenvalues;
- $2N_{\text{c}}$  - number of complex unstable eigenvalues;
- $2N_{\text{im}}^-$  - number neutrally stable eigenvalues of negative Krein signature.

### Definition

Suppose that  $\lambda \in i\mathbb{R}$  is a simple isolated eigenvalue of  $JH''(u_0)$  with the eigenvector  $v$ . Then, the sign of the quadratic form

$$\langle H''(u_0)v, v \rangle_{L^2} = \lambda \langle J^{-1}v, v \rangle_{L^2}$$

is called the Krein signature of the eigenvalue  $\lambda$ .

### III. Massive Thirring Model (MTM)

The nonlinear Dirac equation (MTM) in the space of one dimension are:

$$\begin{cases} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{cases} \quad \text{or} \quad \begin{cases} i\psi_t - \varphi_x - \psi = (\psi^2 + \varphi^2)\bar{\psi}, \\ i\varphi_t + \psi_x + \varphi = (\psi^2 + \varphi^2)\bar{\varphi}. \end{cases}$$

Global solutions exist in  $H^1(\mathbb{R})$  [Goodman *et al.* (2003)]  
or in  $L^2(\mathbb{R})$  [Candy (2011), Huh-Moon (2015)].

Three conserved quantities related to symmetries:

$$Q = \int_{\mathbb{R}} (|u|^2 + |v|^2) dx,$$

$$P = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v}) dx,$$

$$H = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} - v\bar{v}_x + v_x\bar{v}) dx + \int_{\mathbb{R}} (-v\bar{u} - u\bar{v} + 2|u|^2|v|^2) dx,$$

where  $H$  is Hamiltonian. **The quadratic part of  $H$  is sign-indefinite.**

# Existence of solitary waves

Time-periodic space-localized solutions

$$u(x, t) = U_\omega(x)e^{-i\omega t}, \quad v(x, t) = V_\omega(x)e^{-i\omega t}$$

satisfy a system of stationary Dirac equations. They are known in the closed analytic form

$$\begin{cases} u(x, t) = i \sin(\gamma) \operatorname{sech} \left[ x \sin \gamma - i \frac{\gamma}{2} \right] e^{-it \cos \gamma}, \\ v(x, t) = -i \sin(\gamma) \operatorname{sech} \left[ x \sin \gamma + i \frac{\gamma}{2} \right] e^{-it \cos \gamma}. \end{cases}$$

- Translations in  $x$  and  $t$  can be added as free parameters.
- Constraint  $\omega = \cos \gamma \in (-1, 1)$  exists because of the gap in the linear spectrum  $(-\infty, -1] \cup [1, \infty)$ .
- Moving solitons can be obtained from the stationary solitons with the Lorentz transformation.



# Orbital stability of Dirac solitons in $H^1$

The Dirac soliton can not be a constrained minimizer of  $H$ .

However, another higher-order Hamiltonian  $R$  exists in  $H^1(\mathbb{R})$ :

$$R = \int_{\mathbb{R}} \left[ |u_x|^2 + |v_x|^2 - \frac{i}{2}(u_x \bar{u} - \bar{u}_x u)(|u|^2 + 2|v|^2) + \dots \right. \\ \left. - (u\bar{v} + \bar{u}v)(|u|^2 + |v|^2) + 2|u|^2|v|^2(|u|^2 + |v|^2) \right] dx,$$

in addition to the other conserved quantities  $H$ ,  $Q$ , and  $P$ .

## Theorem (P-Shimabukuro (2014))

*There is  $\omega_0 \in (0, 1]$  such that for any fixed  $\omega = \cos \gamma \in (-\omega_0, \omega_0)$ , the Dirac soliton is a local non-degenerate minimizer of  $R$  in  $H^1(\mathbb{R})$  under the constraints of fixed values of  $Q$  and  $P$ .*

## The energy functionals

- Critical points of  $H + \omega Q$  for a fixed  $\omega \in (-1, 1)$  satisfy the stationary MTM equations. After the reduction  $(u, v) = (U, \bar{U})$ , we obtain the first-order equation

$$i \frac{dU}{dx} - \omega U + \bar{U} = 2|U|^2 U.$$

The MTM soliton  $U = U_\omega$  satisfies the first-order equation.

- Critical points of  $R + \Omega Q$  for some fixed  $\Omega \in \mathbb{R}$  satisfy another system of equations. After the reduction  $(u, v) = (U, \bar{U})$ , we obtain the second-order equation

$$\frac{d^2 U}{dx^2} + 6i|U|^2 \frac{dU}{dx} - 6|U|^4 U + 3|U|^2 \bar{U} + U^3 = \Omega U.$$

$U = U_\omega$  also satisfies the second-order equation if  $\Omega = 1 - \omega^2$ .

## The Lyapunov functional for MTM solitons

We define the conserved energy functional in  $H^1(\mathbb{R})$  by

$$\Lambda_\omega := R + (1 - \omega^2)Q, \quad \omega \in (-1, 1),$$

where  $Q = \|u\|_{L^2}^2 + \|v\|_{L^2}^2$ .

- $U_\omega$  is a **critical point** of  $\Lambda_\omega$ .
- The second variation of  $\Lambda_\omega$  can be block-diagonalized

$$S^T \Lambda_\omega''(U_\omega) S = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix},$$

where  $L_+$  and  $L_-$  are  $2 \times 2$  matrix Schrödinger operators.

Chugunova–P (2006); P–Shimabukuro (2014);

- $\Lambda_\omega''(U_\omega)$  has one negative eigenvalue and a double zero eigenvalue for  $\omega > 0$  and  $\omega < 0$ . The zero eigenvalue is quadruple for  $\omega = 0$ .

# Convexity of the energy functional

- Two constraints are added to fix the values of  $Q$  and  $P$ .
- Two constraints are added to eliminate translation and rotation.
- The Hessian operator  $\Lambda''_\omega(U_\omega)$  is strictly positive under the four constraints. The conserved energy functional  $\Lambda_\omega$  becomes convex at  $U_\omega$  in the constrained  $H^1(\mathbb{R})$  space.
- The four constraints can be realized by the choice of four modulation parameters in the soliton orbit:

$$\begin{cases} u(x, t) = i \sin(\gamma) \operatorname{sech} \left[ x \sin(\gamma) - i \frac{\gamma}{2} - \alpha \right] e^{-it \cos(\gamma) - i\beta}, \\ v(x, t) = -i \sin(\gamma) \operatorname{sech} \left[ x \sin(\gamma) + i \frac{\gamma}{2} - \alpha \right] e^{-it \cos(\gamma) - i\beta}, \end{cases}$$

with parameters  $\alpha$ ,  $\beta$ , frequency  $\omega := \cos \gamma$ , and speed  $c$ .

## IV. The defocusing nonlinear Schrödinger equation

The cubic NLS equation

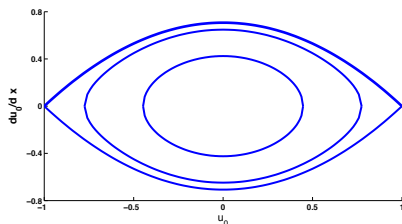
$$i\psi_t + \psi_{xx} - |\psi|^2\psi = 0$$

has long been known for modulational stability of periodic waves.

Periodic waves are of the form  $\psi(x, t) = u_0(x)e^{-it}$ , where

$$u_0''(x) + (1 - |u_0|^2)u_0 = 0$$

has the exact solution  $u_0(x) = \sqrt{1 - \mathcal{E}} \operatorname{sn}\left(x \frac{\sqrt{1+\mathcal{E}}}{\sqrt{2}}; \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}\right)$  with  $\mathcal{E} \in (0, 1)$ .



## Orbital stability of periodic waves in $H_{\text{per}}^1$ or $H_{\text{per}}^2$

Periodic waves are constrained minimizers of energy in  $H_{\text{per}}^1$ :

$$E(\psi) = \int \left[ |\psi_x|^2 + \frac{1}{2}(1 - |\psi|^2)^2 \right] dx$$

under fixed values of

$$Q(\psi) = \int |\psi|^2 dx, \quad M(\psi) = \frac{i}{2} \int (\bar{\psi}\psi_x - \psi\bar{\psi}_x) dx,$$

if the period of perturbations coincides with the period of waves.

[Gallay–Haragus (2007)]

Periodic waves are also constrained minimizers of the higher-order energy

$$R(\psi) = \int \left[ |\psi_{xx}|^2 + 3|\psi|^2|\psi_x|^2 + \frac{1}{2}(\bar{\psi}\psi_x + \psi\bar{\psi}_x)^2 + \frac{1}{2}|\psi|^6 \right] dx,$$

under fixed values of  $Q$  and  $M$  under the same assumption on the period.

## Orbital stability of periodic waves in $H_{\text{Nper}}^2$

Periodic waves are not constrained minimizers of neither  $E$  nor  $R$  if the period of perturbations is multiple to the period of waves.

Nevertheless, there exists a range of values for parameter  $c$  such that the energy functional  $\Lambda_c := R - cE$  is positively definite at  $u_0$ .

[Bottman–Deconinck–Nivala (2011)]

### Theorem (Gallay–P (2015))

For all  $\mathcal{E} \in (0, 1)$ , the second variation of  $\Lambda_c$  at the periodic wave  $u_0$  is nonnegative for perturbations in  $H_{\text{Nper}}^2$  only if  $c \in [c_-, c_+]$  with

$$c_{\pm} := 2 \pm \frac{2\kappa}{1 + \kappa^2}, \quad \kappa = \sqrt{\frac{1 - \mathcal{E}}{1 + \mathcal{E}}}.$$

Moreover, it is strictly positive up to symmetries in  $(c_-, c_+)$  if  $\mathcal{E}$  is small.

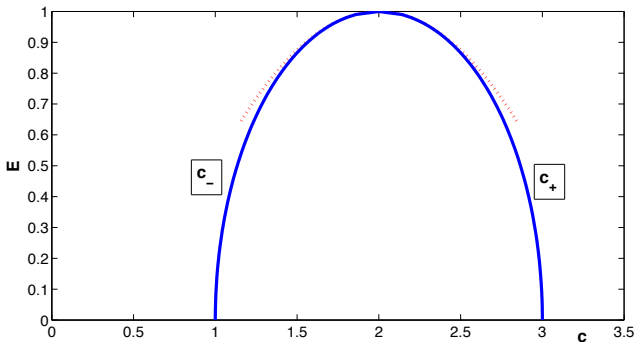


Figure :  $(\mathcal{E}, c)$ -plane for positivity of the second variation of  $\Lambda_c$ .



## A simple perturbative argument

Using the decomposition  $\psi = u_0 + u + iv$  with real-valued perturbation functions  $u$  and  $v$ , we can write

$$\Lambda_c(\psi) - \Lambda_c(u_0) = \langle K_+(c)u, u \rangle_{L^2} + \langle K_-(c)v, v \rangle_{L^2} + \text{cubic terms}$$

where

$$K_+(c)\partial_x u_0 = 0 \quad \text{and} \quad K_-(c)u_0 = 0.$$

If  $u_0 = 0$  (periodic wave of zero amplitude), then

$$\begin{aligned} \langle K_{\pm}(c)u, u \rangle_{L^2} &= \int_{\mathbb{R}} [u_{xx}^2 - cu_x^2 + (c-1)u^2] dx \\ &= \int \left(u_{xx} + \frac{c}{2}u\right)^2 dx - \left(1 - \frac{c}{2}\right)^2 \int u^2 dx. \end{aligned}$$

Then,  $\langle K_{\pm}(c)u, u \rangle_{L^2} \geq 0$  if  $c = 2$ . By perturbative computations, one can find  $(c_-, c_+)$  near  $c = 2$  for  $\mathcal{E} < 1$ .

# Orbital stability of periodic waves in $H_{\text{Nper}}^2$

## Theorem (Gallay–P (2015))

Assume that  $\psi_0 \in H_{\text{Nper}}^2$  and consider the global-in-time solution  $\psi$  to the cubic NLS equation with initial data  $\psi_0$ . For any  $\epsilon > 0$ , there is  $\delta > 0$  s.t. if

$$\|\psi_0 - u_0\|_{H_{\text{Nper}}^2} \leq \delta,$$

then, for any  $t \in \mathbb{R}$ , there exist numbers  $\xi(t)$  and  $\theta(t)$  such that

$$\|e^{i(t+\theta(t))}\psi(\cdot + \xi(t), t) - u_0\|_{H_{\text{Nper}}^2} \leq \epsilon.$$

Moreover,  $\xi, \theta$  are continuous and  $|\dot{\xi}(t)| + |\dot{\theta}(t)| \leq C\epsilon$ .

## V. The Kadomtsev–Petviashvili (KP) equation

The 2D generalization of the KdV equation is the KP equation:

$$(u_t + 6uu_x + u_{xxx})_x = \pm u_{yy},$$

where the plus/minus sign corresponds to KP-I/KP-II equations.

Periodic waves  $u = v(x + ct)$  of the cnoidal form satisfies the 1D KdV equation. Transverse stability is determined for small 2D perturbations  $w$ :

$$(w_t + cw_x + 6(vw)_x + w_{xxx})_x = \pm w_{yy}.$$

**KP-I:** Periodic and solitary waves are transversely unstable [Johnson–Zumbrun (2010); Rousset–Tzvetkov (2011); Hakkaev (2012)]

**KP-II:** Solitary waves are transversely stable [Mizumachi–Tzvetkov (2012); Mizumachi (2015) (2016)]

**KP-II:** Stability of periodic waves is open [Haragus (2010)].

## Conserved quantities for KP-II equation

The momentum of KP-II equation is

$$Q(u) = \int u^2 dx dy$$

The energy of KP-II equation is sign-indefinite near zero:

$$E(u) = \int [u_x^2 - 2u^3 - (\partial_x^{-1} u_y)^2] dx dy.$$

The higher-order energy is still sign-indefinite near zero:

$$R(u) = \int \left[ u_{xx}^2 - 10uu_x^2 + 5u^4 - \frac{10}{3}u_y^2 + \frac{5}{9}(\partial_x^{-2} u_{yy})^2 + \frac{10}{3}u^2 \partial_x^{-2} u_{yy} + \dots \right] dx dy.$$

Molinet–Saut–Tzvetkov (2007)

The previous approach to characterization periodic waves as constrained energy minimizers for a linear combination of  $E(u)$  and  $R(u)$  fails.

## Commuting operators via symplectic operators

1D periodic waves  $u(t, x) = v(x + ct)$  are critical points of  $E(u) + cQ(u)$  with the Hessian operator

$$L_{c,p} = -\partial_x^2 - c - 6v(x) + p^2\partial_x^{-2},$$

where  $p$  is the transverse wave number for the 2D perturbation  $w(x, y) = W(x)e^{ipy}$ .

Search for the commuting self-adjoint operator  $M_{c,p}$  in

$$L_{c,p}\partial_x M_{c,p} = M_{c,p}\partial_x L_{c,p},$$

where  $\partial_x$  defines the symplectic operator for the KP-II equation.

### Theorem (Haragus–Li-P (2017))

*Assume that  $M_{c,p} \geq 0$  and the kernel of  $M_{c,p}$  is contained in the kernel of  $L_{c,p}$ . The spectrum of  $\partial_x L_{c,p}$  is purely imaginary.*

## Algorithmic search of the commuting operator

We are looking for an operator  $M_{c,p}$  to satisfy the commutability relation

$$L_{c,p}\partial_x M_{c,p} = M_{c,p}\partial_x L_{c,p}.$$

Since 1D periodic waves  $u = v(x + ct)$  are also critical points of  $R(u)$ , the Hessian operator  $M_{c,p}$  related to  $R(u)$  satisfies this commutability relation. The operator  $M_{c,p}$  is given by

$$M_{c,p} = \partial_x^4 + 10\partial_x v(x)\partial_x - 10cv(x) - c^2 \\ - \frac{10}{3}p^2 (1 + v(x)\partial_x^{-2} + \partial_x^{-1}v(x)\partial_x^{-1} + \partial_x^{-2}v(x)) + \frac{5}{9}p^4\partial_x^{-4}.$$

### Lemma

*For every  $p \neq 0$ , no value of  $b \in \mathbb{R}$  exists such that  $M_{c,p} - bL_{c,p}$  is positive. Moreover, the number of negative eigenvalues quickly grows in  $L_{N\text{per}}^2$  with larger  $N$ .*

## Algorithmic search of the commuting operator

We are looking for an operator  $M_{c,p}$  to satisfy the commutability relation

$$L_{c,p}\partial_x M_{c,p} = M_{c,p}\partial_x L_{c,p}.$$

By using symbolic computations, we have found another choice of the commuting operator

$$M_{c,p} = \partial_x^4 + 10\partial_x v(x)\partial_x - 10cv(x) - c^2 + \frac{5}{3}p^2(1 + c\partial_x^{-2}).$$

### Lemma

*The operator  $M_{c,p} + 2cL_{c,p}$  is positive in  $L^2_{N\text{per}}$  for every  $p \in \mathbb{R}$  and  $N \in \mathbb{N}$ .*

The periodic travelling wave  $v$  of the KP-II equation is spectrally stable with respect to two-dimensional bounded perturbations.

## Conclusion

- Spectral stability theory is well-developed for relative equilibria in Hamiltonian systems, when the Hessian operators have finitely many negative eigenvalues.
- Orbital stability holds in Hamiltonian systems if the relative equilibrium is a non-degenerate minimum of energy under constraints of fixed mass and momentum.
- For many integrable PDEs (MTM, NLS, KdV), one can use higher-order Hamiltonians to conclude on orbital stability of nonlinear waves.
- For the KP-II equation (in 2D), one can find positive-definite operator unrelated to conserved quantities in order to conclude on spectral stability of nonlinear waves.

The END.