Linear and Nonlinear Dirac Equation: advances and open problems

February 8 – 13, 2017, Como (Italy)

Dipartimento di Scienza e Alta Tecnologia
Università degli Studi dell'Insubria

Programme & Abstracts

Wednesday 08/02/2017

09:50 - 10:10 Opening

10:10 - 11:00 **Piero D'Ancona** – On the nonlinear Dirac equation with an electromagnetic potential

11:00 - 11:30 Coffee Break

11:30 - 12:20 **Timothy Candy** – Large time asymptotics for the Dirac equation

12:20 - 14:30 Lunch Break

14:30 - 15:20 **David Stuart** – Quantization of Maxwell-Dirac in 1+1 dimensions

15:20 - 15:50 **Jean-Claude Cuenin** – Lieb-Thirring type bound for Dirac and fractional Schrödinger operators with complex potentials

15:50 - 16:20 Coffee Break

16:20 - 17:10 **Margherita Nolasco** – An alternative approach to the Dirac operator via the Dirichlet to Neumann operator

17:10 - 18:00 **Gregory Berkolaiko** – Symmetry and Dirac points in the spectrum of honeycomb Laplacian

Thursday 09/02/2017

09:20 - 10:10 **Luis Vega** – A strategy for self-adjointness of Dirac operators: Applications to the MIT bag model and delta-shell interactions

10:10 - 10:40 Coffee Break

10:40 - 11:30 **Naiara Arrizabalaga** – Non-relativistic limit of the MIT bag model

11:30 - 12:20 **Albert Mas** – The relativistic δ-shell interaction in $\mathbb{R}^3$ and its approximation by short range potentials

12:20 - 14:30 Lunch Break

14:30 - 15:20 **Anne-Sophie de Suzzoni** – The relativistic dynamics of an electron coupled with a classical nucleus

15:20 - 15:50 **Federico Cacciafesta** – On the dispersive dynamics of the Dirac equation with critical potentials

15:50 - 16:20 Coffee Break

16:20 - 17:10 **Dmitry Pelinovsky** – Stability of nonlinear waves in integrable Hamiltonian PDEs

Friday 10/02/2017

09:20 - 10:10 **Eric Séré** – Dirac operators with strong Coulomb singularity: domain and min-max levels

10:10 - 10:40 Coffee Break

10:40 - 11:30 **Markus Holzmann** – On the spectral properties of Dirac operators with electrostatic δ-shell interactions

11:30 - 12:20 **Jussi Behrndt** – The spectral shift function for Dirac operators with electrostatic delta-shell interactions

12:20 - 14:00 Lunch Break

14:00 - 14:50 **Nabile Boussaïd** – Nonrelativistic asymptotics of solitary waves in the Dirac equation with Soler-type nonlinearity

14:50 - 15:20 **Sergey Morozov** – On the eigenvalues of perturbed projected Dirac-Coulomb operators

15:20 - 15:50 Coffee Break

15:50 - 16:40 **Andrew Comech** – Stability of solitary waves in the nonlinear Dirac equation

Titles and Abstracts

Non-relativistic limit of the MIT bag model

Naiara Arrizabalaga
UPV/EHU

In this talk we present some spectral asymptotic results of the MIT bag model. This model is the Dirac operator, $-i\alpha \cdot \nabla + m\beta$, defined on a smooth and bounded domain of $\mathbb{R}^3$, Ω, with certain boundary conditions. Specifically, $-i\beta(\alpha \cdot \mathbf{n})\psi = \psi$ must hold at the boundary of Ω, where $\mathbf{n}$ is the outward normal vector and $\psi \in H^1(\Omega, \mathbb{C}^4)$. This model was developed to get a better understanding of the phenomenons involved in the quark-gluon confinement. We study the self-adjointness of the operator and describe the limiting behavior of the eigenvalues of the MIT bag Dirac operator as the mass m tends to $\pm\infty$.

This is a joint work with L. Le Treust and N. Raymond.

The spectral shift function for Dirac operators with electrostatic delta-shell interactions

Jussi Behrndt
Technische Universität Graz

The spectral shift function is a classical notion in perturbation and scattering theory of selfadjoint operators, which has its roots in the works of Lifshitz and Krein. In this talk we shall first review some results on the existence and properties of the spectral shift function. We then discuss a general representation result for the spectral shift function in terms of an abstract Titchmarsh-Weyl m-function and apply this to Dirac operators with electrostatic delta-shell interactions.

This talk is based on a recent joint work with F. Gesztesy and S. Nakamura.

Symmetry and Dirac points in the spectrum of honeycomb Laplacian

Gregory Berkolaiko
Texas A&M University

We will present the symmetry-based reasons for the existence of the Dirac (conical) points in the dispersion relation of the R^2 Schrödinger operator with honeycomb-like potential. We will also discuss the origin of the name of these points. Namely, we will explain results of other researchers (in particular, Fefferman and Weinstein) which show that the dynamics of a wavepacket localized around these points is governed by a 2-dimensional Dirac-type equation.

Nonrelativistic asymptotics of solitary waves in the Dirac equation with Soler-type nonlinearity

Nabile Boussaïd
Laboratoire de Mathématiques de Besançon

We use the perturbation theory to build solitary wave solutions to the nonlinear Dirac equation in any dimension, with the Soler-type nonlinear term which is continuous but not necessarily differentiable. We obtain the asymptotics of solitary waves in the nonrelativistic limit; these asymptotics are important for the linear stability analysis of solitary wave solutions. When the power of the nonlinearity is Schrödinger charge-critical, considering the charge of the corresponding solitary wave in the spirit of the Vakhitov–Kolokolov stability criterion, we show the absence of the degeneracy of zero eigenvalue of the linearization at this solitary wave.

This is a joint work with Andrew Comech (TAMU).

On the dispersive dynamics of the Dirac equation with critical potentials

Federico Cacciafesta
Politecnico di Torino

I will present some recent results concerning weak dispersive estimates for the massless Dirac equation in presence of scaling critical perturbations; in particular, the cases of the Coulomb potential and the Aharonov-Bohm field will be discussed. This talk is based on joint works with E. Séré (univ. Paris Dauphine) and L. Fanelli (univ. "Sapienza" Roma).

Large time asymptotics for the Dirac equation

Timothy Candy
Universität Bielefeld

The question of large time behaviour for the semilinear Dirac equation is particularly delicate due to the lack of good conserved quantities. Thus the current understanding of the asymptotic behaviour is largely restricted to small data regimes. In this talk we will survey recent results on small data linear and nonlinear scattering for two Dirac models, namely the cubic Dirac equation, and the Dirac-Klein-Gordon system.

Stability of solitary waves in the nonlinear Dirac equation

Andrew Comech
Texas A&M University

We study stability of solitary waves in the nonlinear Dirac equation. We consider whether in the nonrelativistic limit (small amplitude solitary waves with frequencies at the end of the spectral gap) the instability of solitary waves could be caused by nonzero-real-part eigenvalues located near the embedded threshold (near $\pm 2mi$). We prove that in the Soler model in spatial dimensions three and below this does not happen due to the presence of $SU(1,1)$ internal symmetry.

The results are obtained together with Nabile Boussaïd, University of Bourgogne Franche-Comte.

Lieb-Thirring type bound for Dirac and fractional Schrödinger operators with complex potentials

Jean-Claude Cuenin
LMU München

We present Lieb-Thirring type bounds for fractional Schrödinger operators and Dirac operators with complex-valued potentials [1]. The spectral bounds hold for discrete as well as for embedded eigenvalues, and we provide counterexamples for the latter showing that our results are essentially sharp. The main technical tools are uniform resolvent estimates in Schatten-Von Neumann classes for the unperturbed operator. These are proved for a large class of kinetic energy operators under a natural curvature assumption on the characteristic set of the corresponding symbol. Estimates of this type have a long tradition in harmonic analysis, and their usefullness for non-selfadjoint spectral problems has first been recognized by R. L. Frank [3]. Going beyond the translation-invariant case, we also consider cases where the unperturbed operator is allowed to have an unbounded electromagnetic background field [2].

References

1. J.-C. Cuenin. Eigenvalue bounds for Dirac and fractional Schrödinger operators with complex potentials. *Journal of Functional Analysis*, 2016 (in print).

2. J.-C. Cuenin and C. E. Kenig. L^p resolvent estimates for magnetic Schrödinger operators with unbounded background fields. *Communications in Partial Differential Equations*, 2016 (in print).

3. Rupert L. Frank. Eigenvalue bounds for Schrödinger operators with complex potentials. *Bull. Lond. Math. Soc.*, 43(4):745–750, 2011.

On the nonlinear Dirac equation with an electromagnetic potential

Piero D'Ancona
Sapienza Università di Roma

In a joint work with M.Okamoto (Shinshu University, Nagano) we prove smoothing and Strichartz estimates for a Dirac equation perturbed by a large potential of critical decay and regularity. In the endpoint case, we prove suitable replacements of these estimates for data of additional angular regularity. As an application we deduce global well posedness and scattering for small data in the energy space with radial symmetry, or with additional angular regularity. Moreover, for a restricted class of potentials, we can extend our results to more general large data under the sole assumption of smallness of the Lochak-Majorana chiral invariants.

The relativistic dynamics of an electron coupled with a classical nucleus

Anne-Sophie de Suzzoni
Université Paris 13

This talk is about the Dirac equation. We consider an electron modeled by a wave function and evolving in the Coulomb field generated by a nucleus. In a very rough way, this should be an equation of the form

$$i\partial_t u = -\Delta u + V(\cdot - q(t))u$$

where u represents the electron while $q(t)$ is the position of the nucleus. When one considers relativistic corrections on the dynamics of an electron, one should replace the Laplacian in the equation by the Dirac operator. Because of limiting processes in the chemistry model from which this is derived, there is also a cubic term in u as a correction in the equation. What is more, the position of the nucleus is also influenced by the dynamics of the electron. Therefore, this equation should be coupled with an equation on q depending on u.

I will present this model and give the first properties of the equation. Then, I will explain why it is well-posed on H^2 with a time of existence depending only on the H^1 norm of the initial datum for u and on the initial datum for q. The linear analysis, namely the properties of the propagator of the equation $i\partial_t u = Du + V(\cdot - q(t))$ where D is the Dirac operator is based on works by Kato, while the non linear analysis is based on a work by Cancs and Lebris.

It is possible to have more than one nucleus. I will explain why.

Joint work with F. Cacciafesta, D. Noja and E. Séré

On the spectral properties of Dirac operators with electrostatic δ-shell interactions

Markus Holzmann
Technische Universität Graz

In this talk I will discuss self-adjointness and basic spectral properties of Dirac operators with electrostatic δ-shell interactions in space dimension $d = 3$. These are self-adjoint operators in $L^2(\mathbb{R}^3; \mathbb{C}^4)$ which are formally given by $A_\eta := A_0 + \eta\delta_\Sigma I_4$, where A_0 is the free Dirac operator, $\eta \in \mathbb{R}$ is the constant interaction strength and $\Sigma \subset \mathbb{R}^3$ is the boundary of a C^2-smooth domain. In order to define these operators in a mathematical rigorous way techniques related to quasi and ordinary boundary triples and their associated Weyl functions are used. After establishing the self-adjointness of A_η, it turns out that some of the spectral properties of A_η are of a different nature, if $\eta = 2c$, where c denotes the speed of light, or $\eta \neq 2c$. In the latter case H^1-smoothness of functions in $\operatorname{dom} A_\eta$ in $\mathbb{R}^3 \setminus \Sigma$ and finiteness of the discrete spectrum are shown and the nonrelativistic limit of A_η, as the speed of light c tends to ∞, is computed. In the case $\eta = 2c$ these properties either differ from those for $\eta \neq 2c$ or they are still unknown.

This talk is based on joint works with J. Behrndt, P. Exner, and V. Lotoreichik.

The relativistic δ-shell interaction in $\mathbb{R}^3$ and its approximation by short range potentials

Albert Mas
Universitat Politècnica de Catalunya

In this talk we will discuss the approximation of a relativistic δ-shell interaction in $\mathbb{R}^3$ by interactions with short range potentials. The relativistic δ-shell interaction is described by the coupling of the free Dirac operator with a singular potential supported on the boundary of a smooth bounded domain Ω, and the short range potentials are given by cutoff functions in a neighbourhood of $\partial\Omega$. During the approximation procedure, we will see how singular integral operators on $\partial\Omega$ come into play and how standard techniques in Calderón-Zygmund theory allow us to develop the approximation mentioned before. In particular, we will talk about the use of maximal operators and pointwise limits almost everywhere to deal with the limit of short range relativistic interactions.

This is a joint work with F. Pizzichillo (BCAM - Basque Center for Applied Mathematics, Bilbao).

On the eigenvalues of perturbed projected Dirac-Coulomb operators

Sergey Morozov
LMU München

Let D be the free Dirac operator in $(0,\infty)$, $\mathbb{R}^2$ or $\mathbb{R}^3$. For $\nu \geq 0$ let D^ν be the distinguished (or, in dimension 1, arbitrary) self-adjoint realisation of $D - \nu/|\cdot|$ and P_+^ν the positive spectral projector of D^ν. For matrix-functions V decaying at infinity we will consider negative eigenvalues of $P_+^\nu(D^\nu - V)P_+^\nu$. We will find out, under which conditions such eigenvalues exist for arbitrary small $V > 0$, estimate the amount of these eigenvalues and sums of their powers. We will also discuss the spectral representation of D^ν and lower bounds on $|D^\nu|$ in terms of fractional Laplacians. The talk is based on resent joint work with David Müller (LMU).

An alternative approach to the Dirac operator via the Dirichlet to Neumann operator

Margherita Nolasco
DISIM-Università dell'Aquila

Using the Foldy – Wouthuysen (unitary) transformation we study problems involving the Dirac operator via elliptic equations in the 4-dim half space $\mathbb{R}^4_+$ with Neumann boundary condition. In particular we give a variational characterization of the eigenvalues and eigenfunctions of the Brown-Ravenhall Hamiltonian and an alternative proof of the Virial theorem for the Dirac operator. We also outline some works in progress and open problems on the Maxwell-Dirac systems for large atoms.

Stability of nonlinear waves in integrable Hamiltonian PDEs

Dmitry Pelinovsky
McMaster University

Many Hamiltonian PDEs (KdV, NLS, KP, MTM) are integrable with the inverse scattering transform method. In particular, they exhibit infinitely many conserved quantities, Backlund-Darboux transformations, solvability with the associated linear systems, and other properties. I will show how to use these properties in order to study the stability problems of nonlinear waves which cannot be solved in general nonintegrable Hamiltonian PDEs. Examples include orbital stability of Dirac solitons in MTM and transverse stability of periodic waves in KP-II.

Dirac operators with strong Coulomb singularity: domain and min-max levels

Eric Séré
Université Paris-Dauphine

This talk reports recent joint work with M.J. Esteban and M. Lewin. We obtain new characterizations of the domain of Dirac operators with strong Coulomb-like potentials, and we prove min-max formulas for the eigenvalues in the spectral gap which generalize earlier results. In particular, we are able to treat the critical case $Z/c = 1$.

Quantization of Maxwell-Dirac in 1+1 dimensions

David Stuart
University of Cambridge

I will discuss the quantization of the Maxwell-Dirac system with periodic spatial boundary conditions, proving an existence theorem via convergence of a lattice approximation and bosonization. The role of gauge invariance in coupling the different sectors in the Lieb-Mattis formalism and the Schwinger anomaly will be explained.

A strategy for self-adjointness of Dirac operators: Applications to the MIT bag model and delta-shell interactions

Luis Vega
UPV/EHU

We develop an approach to prove self-adjointness of Dirac operators with boundary or transmission conditions at a $\mathcal{C}^2$-compact surface without boundary. To do so we are lead to study the layer potential induced by the Dirac system as well as to define traces in a weak sense for functions in the appropriate Sobolev space. Finally, we introduce Calderón projectors associated with the problem and illustrate the method in two special cases: the well-known MIT bag model and an electrostatic δ-shell interaction.